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Passivity-Preserving Model Reduction by Analytic Interpolation

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Passive systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$\lambda(A) \in \mathbb{C}_-$

$D + D^\mathsf{T} > 0$

(A, B) reachable

(C, A) observable

$$\int_0^T u(t)^\mathsf{T} y(t)dt \geq 0 \quad \text{for all } T \geq 0$$

passive

$\Longleftrightarrow$

$$G(s) = C(sI - A)^{-1}B + D$$

positive real

$$G(i\omega) + G(-i\omega)^\mathsf{T} \geq 0, \quad \omega \in \mathbb{R}$$

$$\boxed{G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]}$$

Passivity-preserving model reduction

$$(A, B, C) \longrightarrow (\hat{A}, \hat{B}, \hat{C})$$

$\dim = n$ $\qquad\qquad$ $\dim = r < n$

$$\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + D \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & D \end{array}\right]$$ positive real

Projection method: Find $\quad U, V \in \mathbb{R}^{n \times r}$ such that $\quad U^{\mathsf{T}} V = I_r$

and

$$\hat{A} = U^{\mathsf{T}} A V, \quad \hat{B} = U^{\mathsf{T}} B, \quad \hat{C} = C V$$

Ex: Stochasically balanced model reduction

(ARE) $$AP + PA^{\mathsf{T}} - (B + PC^{\mathsf{T}})(D + D^{\mathsf{T}})^{-1}(B + PC^{\mathsf{T}})^{\mathsf{T}} = 0$$

Solution set: $$P_- \leq P \leq P_+$$

Stochastic balancing: $$TP_-T^{\mathsf{T}} = \Sigma = T^{-\mathsf{T}} P_+^{-1} T^{-1}$$

$\Sigma =$ diagonal matrix consisting of the singular values of $P_- P_+^{-1}$

Truncation: $$U^{\mathsf{T}} = \begin{bmatrix} I_k & 0 \end{bmatrix} T \qquad V = T^{-1} \begin{bmatrix} I_k \\ 0 \end{bmatrix}$$

$$\hat{A} = U^{\mathsf{T}} AV, \quad \hat{B} = U^{\mathsf{T}} B, \quad \hat{C} = CV$$

Antoulas' observation

Consider the class of approximants $\hat{G}(s)$ that retains *r* stable spectral zeros of the original function; i.e., *r* stable zeros of

$$G(s) + G(-s)^{\mathsf{T}}$$

Then, if $s_1, s_2, \ldots, s_r \in \mathbb{C}_+$ are the mirror images (in the imaginary axis) of these spectral zeros, the interpolant

$$\hat{G}(s_j) = G(s_j), \quad j = 1, 2, \ldots, r$$

is positive real. In other words, the passivity property is preserved in such a model reduction procedure.

Recall notation:

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Spectral zeros

$$G(s) + G(-s)^{\mathsf{T}} \sim \left[\begin{array}{cc|c} A & & B \\ & -A^{\mathsf{T}} & -C^{\mathsf{T}} \\ \hline C & B^{\mathsf{T}} & D + D^{\mathsf{T}} \end{array}\right]$$

The spectral zeros are the λ for which the matrix $\mathcal{A} - \lambda\mathcal{E}$ is singular, where

$$\mathcal{A} := \begin{bmatrix} A & & B \\ & -A^{\mathsf{T}} & -C^{\mathsf{T}} \\ C & B^{\mathsf{T}} & D + D^{\mathsf{T}} \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} I_n & & \\ & I_n & \\ & & 0_m \end{bmatrix}$$

Sorensen's algorithm

Partial real Schur decomposition:

$$\begin{bmatrix} A & & B \\ & -A^{\mathsf{T}} & -C^{\mathsf{T}} \\ C & B^{\mathsf{T}} & D + D^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix} R \qquad \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = I_r$$

Singular value decomposition: $Q_x \Sigma^2 {Q_y}^{\mathsf{T}} = X^{\mathsf{T}} Y$

$$\hat{A} = U^{\mathsf{T}} A V, \quad \hat{B} = U^{\mathsf{T}} B, \quad \hat{C} = C V$$

$$V := X Q_x \Sigma^{-1}$$

$$U := Y Q_y \Sigma^{-1}$$

$\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + D$ positive real

Interpolation in the matrix case

Sorensen's solution satisfies

$$\hat{G}(s_j)z_j = G(s_j)z_j, \quad j = 1, \ldots, r$$

where $z_j := Zr_j \neq 0$ with r_j is the right eigenvector of R corresponding to s_j

Moreover, $\hat{G}$ satisfies

$$z_j^{\mathsf{T}}\hat{G}(-s_j) = z_j^{\mathsf{T}}G(-s_j)$$

for each j such that $(-s_jI_k - \hat{A})$ is invertible.

Analytic interpolation with degree constraint

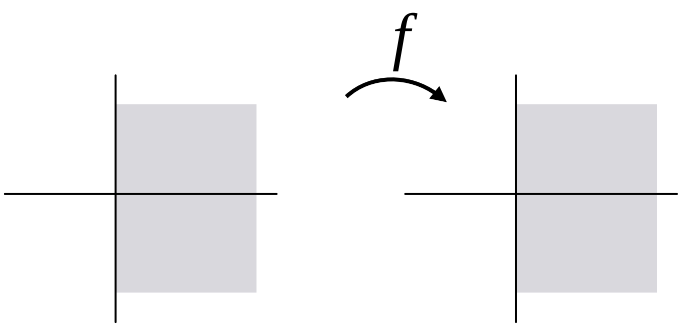
Given: $\{(s_j, w_j) : s_j \in \mathbb{C}_+\}_{j=0}^{r}$

$s_j \neq s_k$ if $j \neq k$, s_0 real

$w_j = \bar{w}_k$ if $s_j = \bar{s}_k$

$$\left[\frac{w_j + \bar{w}_k}{s_j + \bar{s}_k}\right]_{j,k=0}^{r} > 0$$

Find: Positive real function



- f analytic for $\mathrm{Re}\{z\} > 0$
- $\mathrm{Re}\{f(z)\} > 0$ for $\mathrm{Re}\{z\} > 0$

such that (i) $f(s_k) = w_k, \quad k = 0, 1, \ldots, r$

(ii) f rational of degree at most r

Complete parameterization

THEOREM. To each real monic Hurwitz polynomial ρ of degree n there is a unique pair (α, β) of real monic Hurwitz polynomials of degree n such that

(i) $f := \beta/\alpha$ is positive real,

(ii) $f(s_j) = w_j$, $j = 0, 1, \ldots, n$, and

(iii) $\alpha(s)\beta(-s) + \alpha(-s)\beta(s) = \rho(s)\rho(-s)$.

Conversely, any pair (α, β) of real polynomials of degree n satisfying (i) and (ii) determines, via (iii), a unique (modulo sign) Hurwitz polynomial ρ of degree n. The map $\rho \mapsto (\alpha, \beta)$ is a diffeomorphism.

(i) & (iii) $\Longleftrightarrow$ $f(s) + f(-s) = \dfrac{\rho(s)\rho(-s)}{\alpha(s)\alpha(-s)}$

roots of ρ = spectral zeros

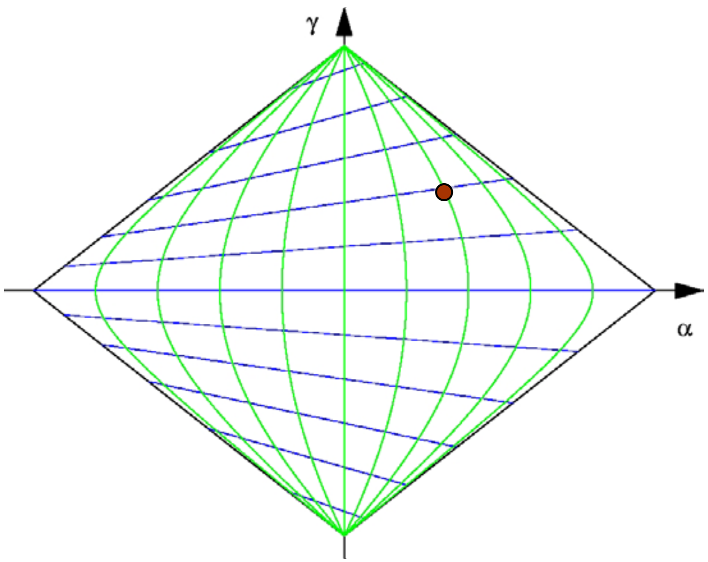
Non-linear coordinates

The manifold of all (α, β) such that $f = \beta/\alpha$ is positive real has two foliations:

A foliation with one leaf for each choice of spectral zeros (Kalman filtering)

Another foliation with one leaf for each choice of $w_0, w_1, \ldots, w_n$

<u>THEOREM</u>. The two foliations intersect transversely so that each leaf in one meets each leaf in the other in exactly one point.



Optimization approach

Given (ρ, w), maximize

$$\int_{-\infty}^{\infty} \left| \frac{\rho(i\omega)}{\tau(i\omega)} \right|^2 \log[f(i\omega) + f(-i\omega)] \frac{d\omega}{\omega^2 + s_0^2}$$

over all positive real f subject to

$$\boxed{\tau(s) := \prod_{j=1}^{n} (s + s_j)}$$

$$f(s_j) = w_j, \quad j = 0, 1, \ldots, n$$

This optimization problem has a unique solution, which has the form

$$f = \frac{\beta}{\alpha} \qquad \text{where} \qquad \alpha(s)\beta(-s) + \alpha(-s)\beta(s) = \rho(s)\rho(-s)$$

determined via the dual problem

Dual problem

Given ρ and any $w \in H^\infty$ such that $w(s_k) = w_k$, $k = 0, 1 \ldots, r$, minimize

$$\int_{-\infty}^{\infty} \left([w(i\omega) + w(-i\omega)] Q(i\omega) - \left| \frac{\rho(i\omega)}{\tau(i\omega)} \right|^2 \log Q(i\omega) \right) \frac{d\omega}{\omega^2 + s_0^2}$$

over all $Q \in \mathfrak{Q}$, where

$$\mathfrak{Q} = \left\{ Q(i\omega) = \operatorname{Re} \sum_{k=0}^{r} \frac{q_k}{i\omega + s_k} \mid Q(i\omega) \geq 0 \right\}.$$

Convex optimization problem with a unique solution

optimal α and
$\alpha(s)\beta(-s) + \alpha(-s)\beta(s) = \rho(s)\rho(-s)$ $\Rightarrow$ $f = \dfrac{\beta}{\alpha}$

Maximum entropy solution

Choose the spectral zeros in the mirror image of the interpolation points; i.e.,

$$\rho(s) = \tau(s)^*.$$

Then the primal problem amounts to maximize

$$\int_{-\infty}^{\infty} \log[f(i\omega) + f(-i\omega)] \frac{d\omega}{\omega^2 + s_0^2}$$

over all positive real f satisfying the interpolation constraints.

linear problem   Cf. Mustafa-Glover

Back to the dual problem

$$\mathbb{J}_P(Q) := \int_{-\infty}^{\infty} [\Phi Q - P \log Q] \frac{d\omega}{\omega^2 + s_0^2} \longrightarrow \min$$

$$\Phi(s) := \frac{1}{2}\left[G(s) + G(-s)^{\mathsf{T}}\right], \quad P(i\omega) := \left|\frac{\rho(i\omega)}{\tau(i\omega)}\right|^2$$

- Maximum entropy solution for $\rho(s) = \tau(s)^*$ ➡ $P \equiv 1$
- The Antoulas-Sorensen solution also requires choosing interpolation points in zeros of Φ plus $s = \infty$

The Antoulas-Sorensen method as the maximum entropy solution

$$\begin{aligned}\hat{G}(\infty) &= w_0 := D \\ \hat{G}(s_k) &= w_k := G(s_k), \quad k = 1, 2, \ldots, r\end{aligned}$$

where $s_1, s_2, \ldots, s_r$ chosen in the mirror image of a self-conjugate set of spectral zeros.

THEOREM. For $s_0 > 0$ sufficiently large, let f_{s_0} be the maximum entropy solution corresponding to the interpolation conditions

$$f_{s_0}(s_0) = w_0, \quad f_{s_0}(s_k) = w_k, \quad k = 1, \ldots, r.$$

Then, as $s_0 \to \infty$, $f_{s_0} \to \hat{G}$ pointwise (except in the poles of $\hat{G}$).

$$\boxed{\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + D}$$

$$\hat{C} = (\tilde{P}^{-1}w)^{\mathsf{T}} \qquad \tilde{P} = \left[\frac{w_j + \bar{w}_\ell}{s_j + \bar{s}_\ell}\right]_{j,\ell=1}^{r} \qquad w = \begin{bmatrix} w_1 - w_0 \\ w_2 - w_0 \\ \vdots \\ w_r - w_0 \end{bmatrix}$$

$$\hat{A} = -\Lambda + h\hat{C} \qquad \Lambda := \operatorname{diag}(s_1, s_2, \ldots, s_r)$$

$$h = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\hat{B} = 2w_0(Q\hat{C}^* + h)$$

$$\hat{A}Q + Q\hat{A}^* + hh^* = 0$$

Global-analysis approach

Since we have a smooth parameterization, the reduced-order solution obtained by the (numerically efficent) Sorensen algorithm (or some other method) can be tuned to specifications by moving the

- spectral zeros
- interpolation points

while passivity and degree are preserved.

A benchmark problem

Original system:

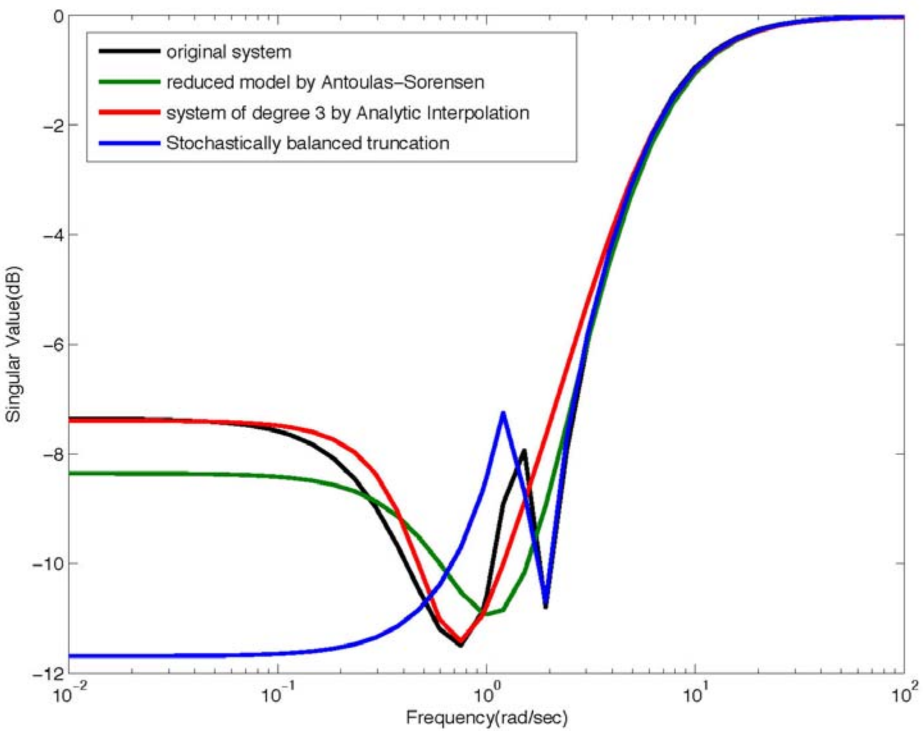
$$G(s) = \frac{s^5 + 3s^4 + 6s^3 + 9s^2 + 7s + 3}{s^5 + 7s^4 + 14s^3 + 21s^2 + 23s + 7}$$

Antoulas-Sorensen:

$$\hat{G}(s) = \frac{s^3 + 2.553s^2 + 2.906s + 1.173}{s^3 + 6.681s^2 + 8.459s + 3.07}$$

Global-analysis approach:

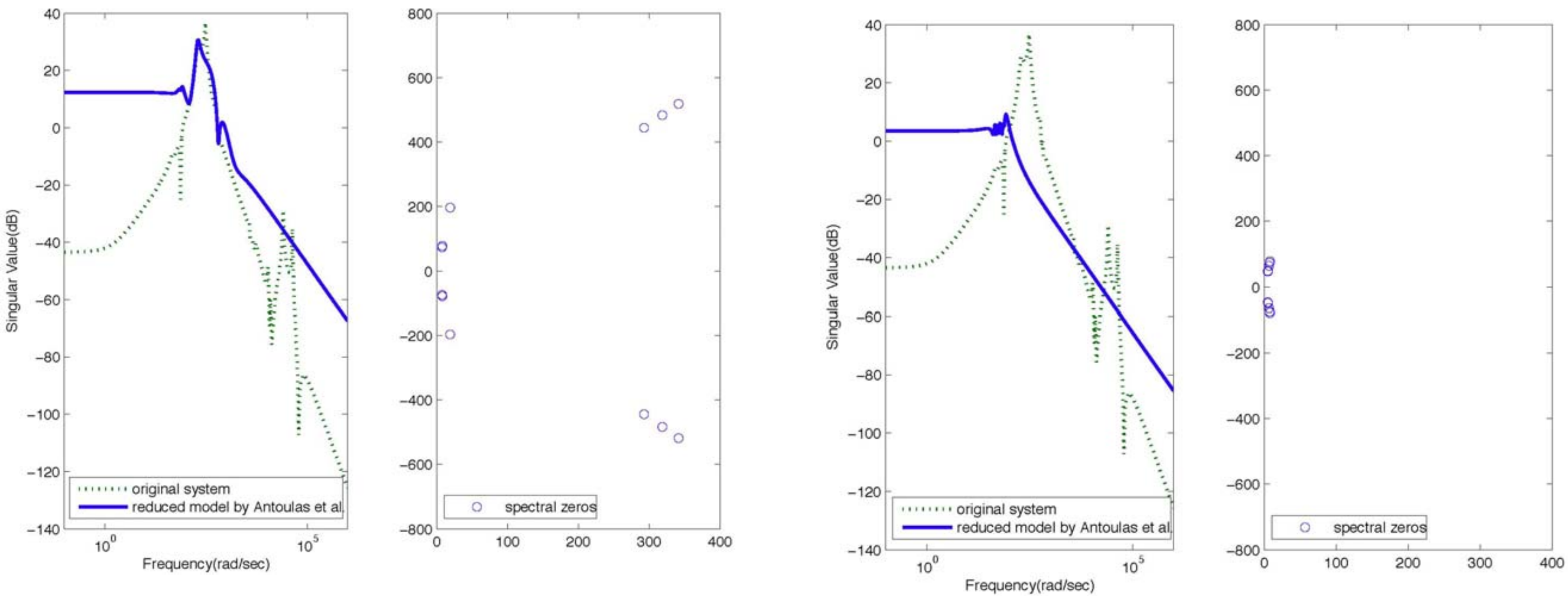
$$f(s) = \frac{1.002s^3 + 2.84s^2 + 1.927s + 0.8978}{s^3 + 7.298s^2 + 6.084s + 2.099}$$



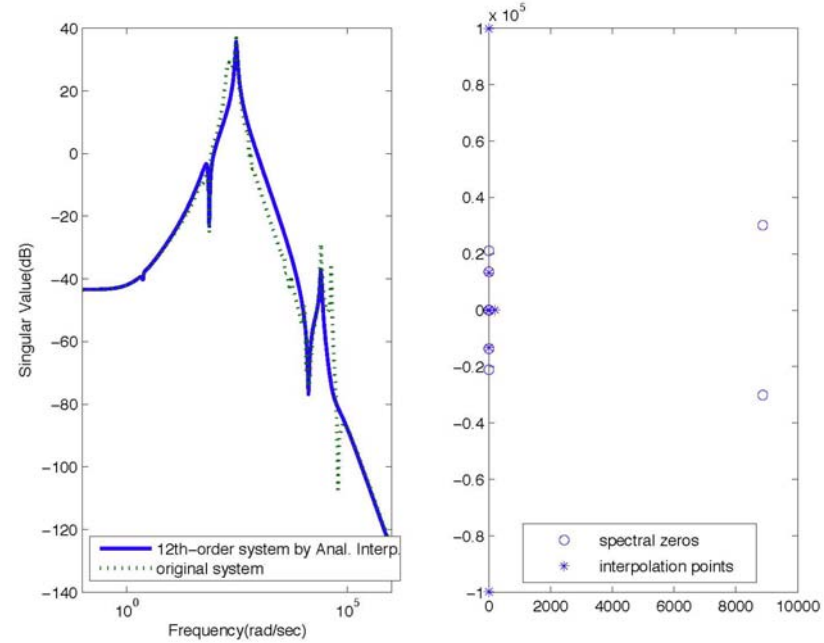
A large-scale problem: A CD player

Model reduction: $\deg G = 120 \longrightarrow \deg \hat{G} = 12$

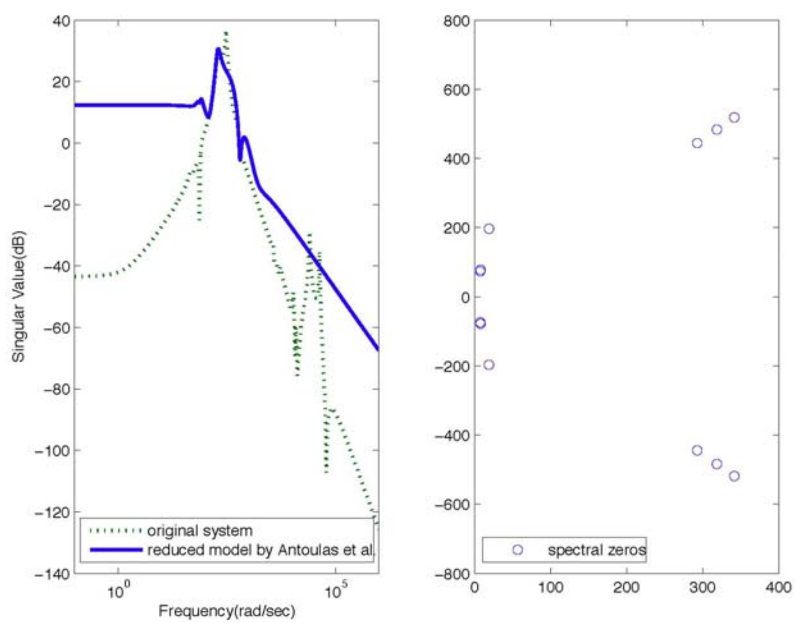
Antoulas-Sorensen solutions:

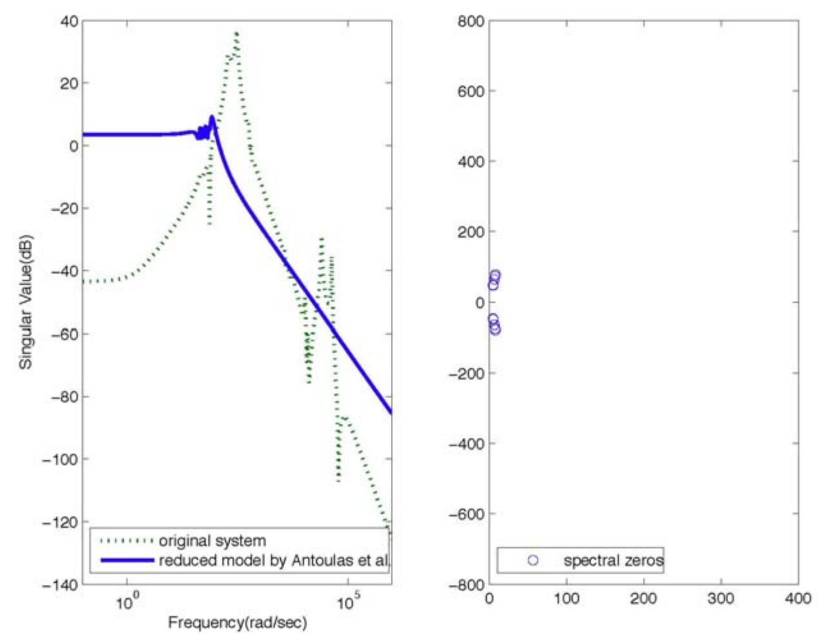


Global-analysis solution



Antoulas-Sorensen solutions





Discrete-time NP interpolation with degree constraint

Given:

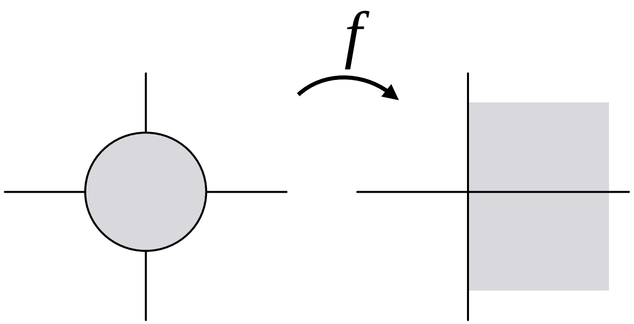
$z_0,\ z_1,\ \ldots,\ z_n$ $\quad |z_k| < 1$ (distinct)

$w_0,\ w_1,\ \ldots,\ w_n$ $\quad \mathrm{Re}\{w_k\} > 0$

such that Pick matrix

$$P_n = \left[\frac{w_k + \bar{w}_\ell}{1 - z_k \bar{z}_\ell} \right]_{k,\ell=0}^{n} > 0$$

Find: Carathéodory function



- f analytic for $|z| \leq 1$
- $\mathrm{Re}\{f(z)\} > 0$ for $|z| \leq 1$

$\boxed{f \in \mathcal{C}_+}$

such that (i) $\quad f(z_k) = w_k, \quad k = 0, 1, \ldots, n$

(ii) $\quad f$ rational of degree at most n

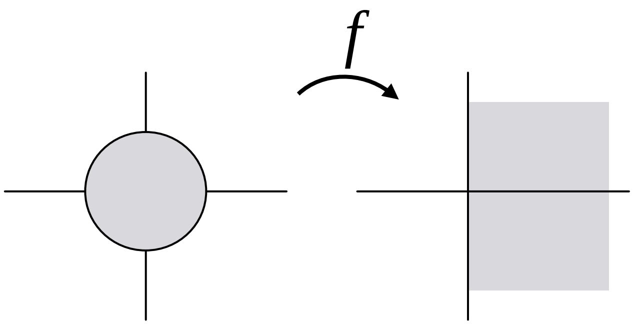
For simplicity normalize:

$(z_0, w_0) = (0, 1)$

$$f = \frac{\beta}{\alpha}, \qquad \alpha, \beta \in \mathcal{S}_n$$ Schur polynomials of degree n

$$\boxed{\mathcal{P}_n = \left\{ (\alpha, \beta) \in \mathcal{S}_n \times \mathcal{S}_n \mid f = \frac{\beta}{\alpha} \in \mathcal{C}_+ \right\}}$$

$$f \in \mathcal{C}_+ \Rightarrow \operatorname{Re} f = \frac{1}{2}\left(\frac{\beta}{\alpha} + \frac{\beta^*}{\alpha^*}\right) = \frac{1}{2}\frac{\alpha^*\beta + \alpha\beta^*}{\alpha\alpha^*} > 0$$



$$\frac{1}{2}(\alpha^*\beta + \alpha\beta^*) = \rho\rho^*, \qquad \rho \in \mathcal{S}_n$$ roots are spectral zeros

$$\boxed{\operatorname{Re}(\alpha^*\beta) = |\rho|^2}$$

Complete parameterization

$$\mathcal{P}_n = \left\{ (\alpha, \beta) \in \mathcal{S}_n \times \mathcal{S}_n \mid f = \frac{\beta}{\alpha} \in \mathcal{C}_+ \right\}$$

THEOREM. To each monic $\rho \in \mathcal{S}_n$ there is a unique pair $(\alpha, \beta) \in \mathcal{P}_n$ such that $f = \beta/\alpha$ satisfies

(i) $f(z_k) = w_k, \quad k = 0, 1, \ldots, n$

(ii) $\mathrm{Re}(\alpha^* \beta) = |\rho|^2$

The correspondence $\rho \leftrightarrow \alpha$ is a diffeomorphism, which can be extended to the boundary as a homeomorphism.

Optimization approach

Given (ρ, w), maximize

$$\int_{-\pi}^{\pi} \left| \frac{\rho(e^{i\theta})}{\tau(e^{i\theta})} \right|^2 \log \left[\operatorname{Re} f(e^{i\theta}) \right] d\theta$$

$$\boxed{\tau(z) = \prod_{k=0}^{n} (1 - \bar{z}_k z)}$$

over all $f \in \mathcal{C}_+$ subject to

$$f(z_k) = w_k, \quad k = 0, 1, \ldots, n$$

This optimization problem has a unique solution, which has the form

$$f = \frac{\beta}{\alpha}, \qquad \alpha, \beta \in \mathcal{S}_n \quad \text{where} \quad \operatorname{Re}(\alpha^* \beta) = |\rho|^2$$

determined via the dual problem:

Dual problem

$$\tau(z) = \prod_{k=1}^{n} (1 - \bar{z}_k z)$$

Given (ρ, w), minimize the strictly convex functional

$$\mathbb{J}_\rho(q) = \mathrm{Re} \sum_{k=0}^{n} w_k q_k - \int_{-\pi}^{\pi} \left| \frac{\rho(e^{i\theta})}{\tau(e^{i\theta})} \right|^2 \log Q(e^{i\theta}) \frac{d\theta}{2\pi}$$

over the convex set of all $(q_0, q_1, \dots, q_n)$ such that

$$Q(e^{i\theta}) := \mathrm{Re} \left(\sum_{k=0}^{n} \frac{q_k}{1 - \bar{z}_k e^{i\theta}} \right) > 0, \quad \text{for all } \theta \in [-\pi, \pi]$$

THEOREM. There is a unique minimum.

Then $f = \dfrac{\beta}{\alpha}$, $\alpha, \beta \in \mathcal{S}_n$ where $\left| \dfrac{\alpha(e^{i\theta})}{\tau(e^{i\theta})} \right|^2 = \hat{Q}(e^{i\theta})$

and $\mathrm{Re}(\alpha^* \beta) = |\rho|^2$

Primal problem reformulated

$$\mathbb{I}_P(\hat{\Phi}) := \int_{-\pi}^{\pi} P \log \hat{\Phi} \frac{d\theta}{2\pi} \longrightarrow \max$$

$$\hat{\Phi}(z) := \frac{1}{2}\left[\hat{G}(z) + \hat{G}(z^{-1})^{\mathsf{T}}\right], \quad P(e^{i\theta}) := \left|\frac{\rho(e^{i\theta})}{\tau(e^{i\theta})}\right|^2$$

Optimal solution: $\hat{\Phi}(z) = \dfrac{P(z)}{Q(z)}$ where Q solution of the dual problem

Maximum entropy solution for $P \equiv 1$

Dual problem reformulated

$$\mathbb{J}_P(Q) := \int_{-\pi}^{\pi} [\Phi Q - P \log Q] \frac{d\theta}{2\pi} \longrightarrow \min$$

$$\Phi(z) := \frac{1}{2}\left[G(z) + G(z^{-1})^{\mathsf{T}}\right], \quad P(e^{i\theta}) := \left|\frac{\rho(e^{i\theta})}{\tau(e^{i\theta})}\right|^2$$

$$Q(e^{i\theta}) = \mathrm{Re}\sum_{k=0}^{r} q_k g_k(e^{i\theta}) \geq 0 \quad \text{where} \quad g_k(z) := \frac{1}{1 - \bar{z}_k z}$$

For $P \equiv 1$, the optimal solution: $\hat{\Phi}^{-1}$

where $\hat{\Phi}$ optimal solution of primal problem

Kullback-Leibler divergence

$$D(y\|\hat{y}) := \limsup_{N\to\infty} \frac{1}{N} \int_{\mathbb{R}^n} p_y^N(x) \log \frac{p_y^N(x)}{p_{\hat{y}}^N(x)}\, dx$$

If the stationary processes y and $\hat{y}$ have spectral densities Φ and $\hat{\Phi}$, respectively, then

$$D(y\|z) = \frac{1}{2} \int_{-\pi}^{\pi} \left[(\Phi - \hat{\Phi})\hat{\Phi}^{-1} - \log(\Phi\hat{\Phi}^{-1}) \right] \frac{d\theta}{2\pi}$$

prediction-error approximation

Anderson, Moore and Hawkes
Stoorvogel, van Schuppen

Consequently,

$$D(y\|z) = \frac{1}{2}\mathbb{J}(\hat{\Phi}^{-1}) - \frac{1}{2}\underbrace{\left[1 + \int_{-\pi}^{\pi} \log\Phi \frac{d\theta}{2\pi}\right]}_{\text{constant}}$$

The maximum entropy solution $\hat{\Phi}$ is the minimum prediction-error approximant in the model class

$$\hat{\Phi}^{-1}(z) = \mathrm{Re}\sum_{k=0}^{r} q_k g_k(e^{i\theta}) \geq 0 \qquad g_k(z) := \frac{1}{1-\bar{z}_k z}$$

The Antoulas-Sorensen approach corresponds to the choice of basis functions in which $z_1, z_2, \ldots, z_r$ are spectral zeros.

Conclusions

- The Antoulas-Sorensen solution is essentially
  - the maximum entropy solution
  - the minimum prediction-error solution in a model class with spectral zeros at spectral zeros of the function to be approximated
- It can in general be improved by smooth tuning of the spectral zeros and the interpolation points