Symmetric Darlington Synthesis: a frequency domain approach for the real case

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About the title...

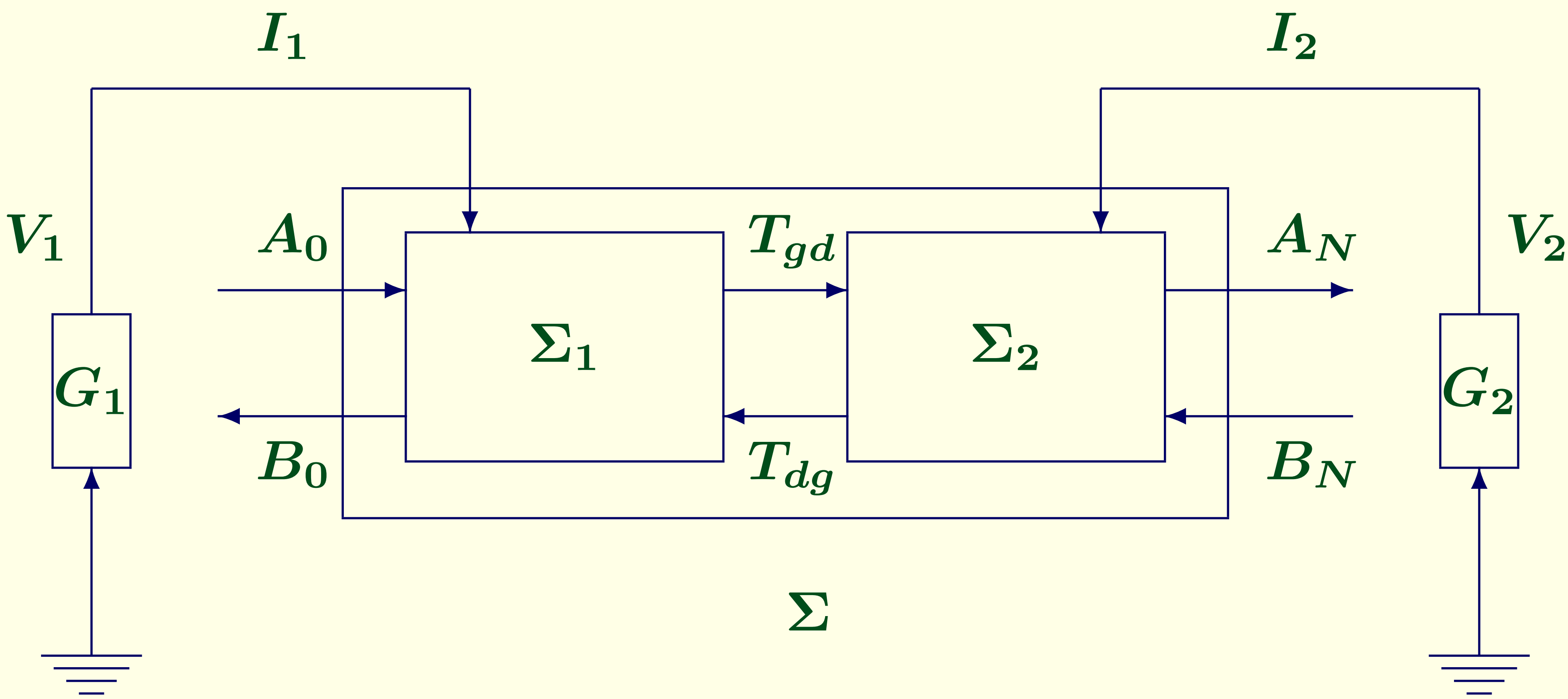
Darlington Synthesis is old (1939), and was mainly studied in a system theoretic framework in the '70 and later (Anderson, Vongpaniltlerd, Dewilde and others) but it occurs in some new problems (like mobile phones SAW filters)

Some physical constraints in these filters make the optimal tuning an interesting mathematical problem where Darlington synthesis plays a crucial role.

In this application it's important to solve the real problem!

Surface Acoustic Wave Filters

The filter is constituted of two transducers Σ_1 and Σ_2.



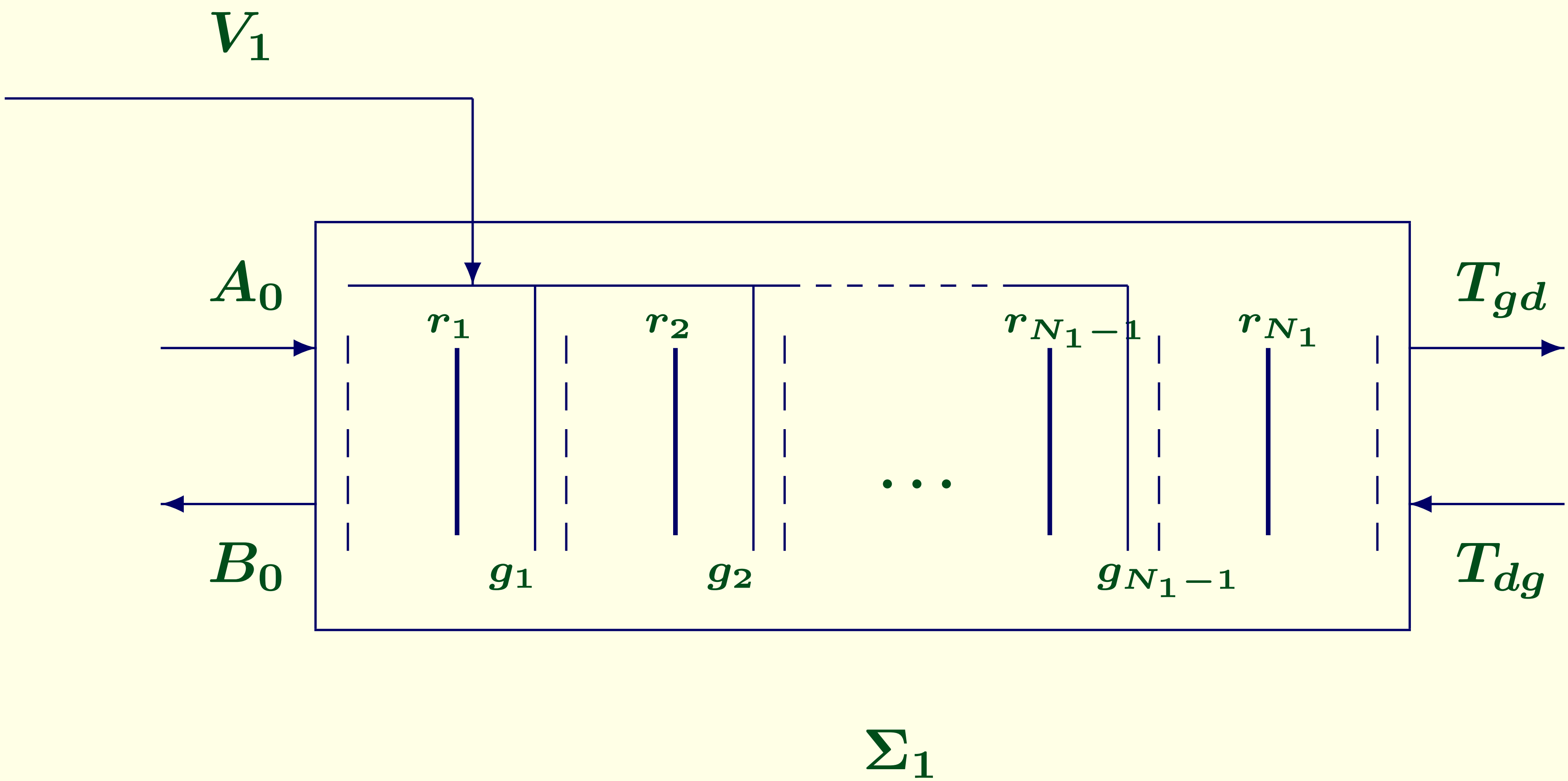


Figure 1: The left transducer Σ_1.

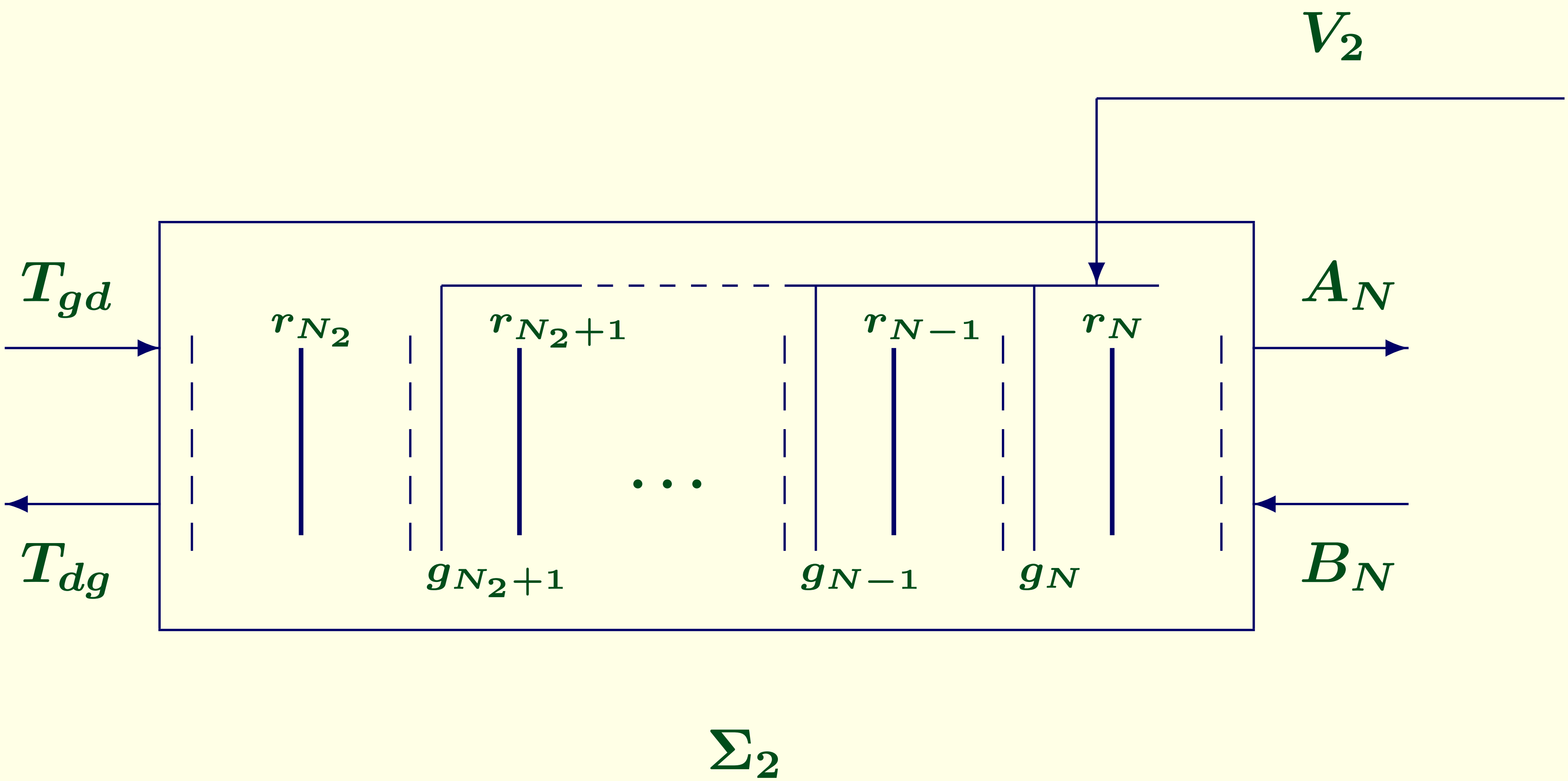


Figure 2: The right transducer Σ_2.

MAIN PROBLEMS IN SYMMETRIC DARLINGTON SYNTHESIS

a) Size constraint.

Given S $p \times p$ symmetric and Schur (13) of deg n, find a $2p \times 2p$ inner (14) extension

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} \Bigg\} 2p$$

which is symmetric and of minimal degree.

b) Degree constraint

Given S $p \times p$ symmetric and Schur of deg n, find a $(2p+q) \times (2p+q)$ inner extension

$$\mathcal{S} = \left[\begin{array}{cc} S_{11} & S_{12} \\ S_{21} & S \end{array} \right] \Bigg\} 2p+q$$

which is symmetric and of same degree n.

Remarks

- Problem a) in general will have no solution smaller than p (and, in fact, we need S strictly contractive). For the complex case a solution which uses state space tools is already known (BEGO 2006, submitted).
- Problem b) is known to have a solution (Vongpanitlerd 1970, Anderson-Vongpanitlerd 1973) of degree $2p + n$.
- Since we ask S to be symmetric and not hermitian, we should expect different results if we require S to be have real coefficients.

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OUTLINE OF THE TALK

- **Central Idea**
- **Scalar example**
- **A symmetric extension of Symmetric Schur functions of the same size**
- **A symmetric extension of Symmetric Schur functions of the same degree**
- **The real case**

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Central Idea

If $\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ **is inner (and** S_{22} **strictly contractive),**

than

$$S_{21}^{-1} S_{12}^{T}$$

is all pass; and also,

$$\det \mathcal{S} = -\det S_{12} \det S_{21}^{-*}$$

Example: suppose S is scalar Schur; we look for S_{11}, S_{12}, S_{21} such that

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix}$$

is symmetric and inner

Theorem 1 *There exists a symmetric inner extension of S of the same degree if and only if $1 - SS^*$ has only zeros of even mutiplicities.*

Proof. Let us write $\mathcal{S}$ as

$$\begin{bmatrix} \frac{p_{11}}{q} & \frac{p_{12}}{q} \\ \frac{p_{21}}{q} & \frac{p_{22}}{q} \end{bmatrix}$$

Then $\mathcal{S}$ inner implies $\mathcal{S}^* = \mathcal{S}^{-1}$ i.e.

$$\frac{1}{q^*}\begin{bmatrix} p_{11}^* & p_{21}^* \\ p_{12}^* & p_{22}^* \end{bmatrix} = \frac{q}{q^*}\frac{1}{q}\begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix}$$

which yields:

$$p_{11} = p_{22}^* \qquad p_{12} = -p_{21}^*$$

and in view of symmetry we get

$$p_{21} = -p_{21}^*$$

i.e.

$$qq^*(I - SS^*) = qq^* - p_{22}p_{22}^* = p_{21}p_{21}^* = p_{21}^2$$

as wanted. ■

But we can always constuct a higher degree extension.

Theorem 2 *Assume now $S = \frac{s}{q}$; then $qq^* - ss^*$ will admit the factorization*

$$qq^* - ss^* = (r_1 r_1^*)^2 r_2 r_2^*$$

with r_2 stable and with simple zeros. Then

$$\begin{bmatrix} \frac{-s^*}{q} \frac{r_2^*}{r_2} & \frac{(r_1 r_1^*) r_2^*}{q} \\ \frac{(r_1 r_1^*) r_2^*}{q} & \frac{s}{q} \end{bmatrix}$$

is a minimal degree symmetric inner extension of S.

Proof. It's obviously symmetric; we only show it's inner. But this is easy. In fact:

$$\begin{bmatrix} \frac{-s^*}{q}\frac{r_2^*}{r_2} & \frac{(r_1 r_1^*) r_2^*}{q} \\ \frac{(r_1 r_1^*) r_2^*}{q} & \frac{s}{q} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s^*}{q} & \frac{(r_1 r_1^*) r_2^*}{q} \\ \frac{(-r_1 r_1^*) r_2}{q} & \frac{s}{q} \end{bmatrix} \begin{bmatrix} \frac{-r_2^*}{r_2} & 0 \\ 0 & 1 \end{bmatrix}$$

■

So, a symmetric inner extension is not trivial even in the scalar case.

Are things much more difficult in the multivariable case?

In fact, not really: in the scalar case we were looking at double zeros of $1 - SS^*$.

In the mutivariable case, surprisingly, we just need to look for double zeros of $\det(I - SS^*)$!

Strategy

- **Construct a symmetric extension $\mathcal{S}$ of dimension $2p \times 2p$ and degree $2n$ (it's well known it's always possible).**
- **Peel away double zeros of $\det(I - SS^*)$ from the extension to reduce the degree.**
- **Show minimality when there are no double zeros left.**
- **Add one dimension to $\mathcal{S}$ to get an extension of degree n.**
- **Extend (with some changes) to the real case.**

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Tools

All quite simple...

- **Inner completion of a wide inner: easy in state space. But need a frequency domain expression!**
- **A symmetric constant matrix T (of $\dim > 1$) always has a complex solution v to**

$$v^T T v = 0$$

  if T and v are real, must look at the signature of M.

- **If M is symmetric inner and has factorizations**

$$M = Q_1^T M_1 Q_1 \qquad M = Q_2^T M_2 Q_2$$

  then $M = Q_3^T M_3 Q$, where Q is the left least common multiple of Q_1, Q_2.

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Remark. We can assume, without loss of generality, that $(I + S)$ is invertible in the closed right half-plane.

Symmetric extension of degree $2n$

For the minimum-phase solution S_{21} of

$$S_{21}S_{21}^{*} = I - SS^{*}$$

we construct a square extension of the same McMillan degree:

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix}$$

Lemma 3 *Define*

$$\mathcal{S}_0 := \begin{bmatrix} S_{11} S_{21}^{-1} S_{12}^T & S_{12} \\ S_{12}^T & S \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} \begin{bmatrix} S_{21}^{-1} S_{12}^T & 0 \\ 0 & I \end{bmatrix}$$

then $\mathcal{S}_0$ is symmetric and inner of degree $2n$.

Proof. We need to show that

- $Q := S_{21}^{-1} S_{12}^T$ is inner of degree n and
- the $(1,1)$ term od $\mathcal{S}_0$, which is $S_{11} S_{21}^{-1} S_{12}^T$, is symmetric.

$Q = S_{21}^{-1} S_{12}^T$ is all pass: in view of symmetry

$$(I - S^* S)^T = I - SS^*$$

or

$$(S_{12}^* S_{12})^T = S_{21} S_{21}^*$$

and

$$S_{21}^{-1} S_{12}^T (S_{12}^*)^T (S_{21}^*)^{-1} = QQ^* = I$$

Since S_{21} is outer, Q is inner.

To show that Q has degree n and is symmetric, we need

Theorem 16: if $\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix}$ is a minimal degree inner extension of $[S_{21}, S]$, then the following DSS factorization holds

$$S_{12}(I+S)^{-1} = -MS_{21}^{*}(I+S^{*})^{-1} \tag{1}$$

and it's also:

$$S_{11} = M - S_{12}(I+S)^{-1}S_{21}$$

(want to see Theorem 16?)

Now Q is stable and

$$Q = S_{21}^{-1} S_{12}^{T} = \overbrace{\underbrace{S_{21}^{-1}(I+S)}_{\substack{\text{stable} \\ \text{degree } n}} \underbrace{(I+S^{*})^{-1}(S_{21}^{*})^{T}}_{\substack{\text{antistable} \\ \text{degree } n}}}^{\text{all-pass}} \overbrace{M^{T}}^{\substack{\text{degree } n \\ \text{inner}}}$$

so Q has degree n.

the $(1,1)$ term of S_0, $S_{11}Q$ is symmetric.

$$\begin{aligned} S_{11}Q &= [M + S_{12}(I+S)^{-1}S_{21}]S_{21}^{-1}S_{12}^T \\ &= M\,\underbrace{S_{21}^{-1}(I+S)}_{\beta^{-1}}\,\underbrace{(I+S^*)^{-1}(S_{21}^*)^T}_{(\beta^*)^T}\,M^T \\ &\quad + S_{12}(I+S)^{-1}S_{12}^T \end{aligned}$$

Since $T := \beta^{-1}(\beta^*)^T$ is all-pass, it is symmetric

$$T^* = (\beta^{-1}(\beta^*)^T)^* = \beta^T(\beta^{-1})^* = [(\beta^*)^T]^{-1}\beta = T^{-1}$$

as wanted.

T is called phase function in stochastic realization.

In conclusion, $\mathcal{S}_0$ is a symmetric extension of degree $2n$. Well known! (see Anderson-Vongpanitlerd 1973)

What about minimal dimension?

Theorem 4 *Let S_{21} be the outer factor of $I - SS^*$ and let*

$$\mathcal{S}_0 = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} \begin{bmatrix} S_{21}^{-1} S_{12}^T & 0 \\ 0 & I \end{bmatrix}$$

*be the symmetric inner extension of the wide inner function $[S_{21}, S]$. Then there exists an inner symmetric extension $\mathcal{S}$ of S of $n + k$ where k is the number of **odd** zeros of*

$$\phi(z) := \det Q(z) = \det S_{21}^{-1}(z) S_{12}^T(z)$$

that is, if $r_1, r_{2,}, \ldots, r_m$ are the multiplicities of the zeros of ϕ, we set

$$k := \sum_{i=1}^{m} (r_i \mod 2)$$

The idea is that, if $\det Q$ has a double zero, we can reduce the degree of $\mathcal{S}_0$ by 2.

HOW?

Find an inner function R of maximal degree such that

$$\mathcal{S} := \begin{bmatrix} (R^*)^T & 0 \\ 0 & I \end{bmatrix} \mathcal{S}_0 \begin{bmatrix} R^* & 0 \\ 0 & I \end{bmatrix}$$

is analytic. Then $\mathcal{S}$ will be a symmetric, inner extension of S of minimal degree.

TWO (SPECIAL) CASES!

We look at these because we can always multiply by a unitary function and its transposed. Let $T(s)$ be $q \times q$ symmetric and let

$$b_\omega(z) := \frac{1 - z\bar{\omega}}{z - \omega}$$

1) Geometric muliplicity 2: Suppose $T(s)$ is divisible on the right by

$$\begin{bmatrix} (b_\omega^*(z))^2 & 0 \\ 0 & I_{q-1} \end{bmatrix}$$

Then

$$T = \begin{bmatrix} b_\omega^2 t_{11} & b_\omega^2 T_{12} \\ b_\omega^2 T_{21} & T_{22} \end{bmatrix}$$

so that

$$\begin{bmatrix} b_\omega^* & 0 \\ 0 & I_{q-1} \end{bmatrix} \begin{bmatrix} b_\omega^2 t_{11} & b_\omega^2 T_{12} \\ b_\omega^2 T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} b_\omega^* & 0 \\ 0 & I_{q-1} \end{bmatrix}$$

is analytic.

2) Suppose $T(s)$ is divisible on the right by

$$\begin{bmatrix} b^*_\omega(z) & 0 & 0 \\ 0 & b^*_\omega(z) & 0 \\ 0 & 0 & I_{q-2} \end{bmatrix}$$

Then

$$T(z) = \begin{bmatrix} b_\omega(z)T_{11}(z) & b_\omega(z)T_{12}(z) \\ b_\omega(z)T_{21}(z) & T_{22}(z) \end{bmatrix}$$

where T_{11} is 2×2

Previous idea does not work; but, if H is symmetric, we can write it as

$$H = U^T \Lambda^2 U$$

with U unitary and Λ positive diagonal (Takagi decomposition). We can thus assume that

$$T_{11}(\omega) = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

Let now $v := \begin{bmatrix} \lambda_2 & i\lambda_1 \end{bmatrix}$

Then

$$vT_{11}(\omega)v^T = \begin{bmatrix} \lambda_2 & i\lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ i\lambda_1 \end{bmatrix} = 0$$

Thus define the Blascke factor R_ω as follows: let P_v be the projection matrix onto the span of v and

$$R_\omega(z) := b_\omega(z)P_v^* + I - P_v^*$$

Then

$$P_v T_{11} P_v^T = 0$$

and thus

$$(R_\omega^*(z))^T b_\omega(z) T_{11}(z) R_\omega^*(z)$$

is analytic in ω, as wanted.

Thus we can reduce the degree of $\mathcal{S}$ by 2 as long as we have double zeros. The process ends when all the zeros are simple.

Minimality

Theorem 4 claims that we can get an extension of degree $n + \kappa$. We show now it has minimal degree.

Proposition 5 ***All rational inner extensions of a contractive rational function S, can be written in the form***

$$\begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix} \mathcal{S} \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}$$

where L, R and $\mathcal{S}$ are inner, and $\mathcal{S}$ is a minimal degree inner extension of S.

The second result we need is:

Lemma 6 *Let $\mathcal{S}$ and be minimal symmetric extension of S. Define*

$$\phi_{\check{\mathcal{S}}}(s) := \det \check{\mathcal{S}}_{21}^{-1}(s)\check{\mathcal{S}}_{12}(s)$$

and let κ denote the number of distinct zeros of $\phi_{\check{\mathcal{S}}}$ with odd multiplicity. Then

$$\phi_S := \det S_{21}^{-1}(s)S_{12}(s)$$

has degree greater than or equal to κ.

Theorem 7 *Let S be a symmetric Schur function, strictly contractive at infinity and let* $\check{\mathcal{S}} = \begin{bmatrix} \check{\mathcal{S}}_{11} & \check{\mathcal{S}}_{12} \\ \check{\mathcal{S}}_{21} & S \end{bmatrix}$ *be its minimal extension with $\check{\mathcal{S}}_{21}$ outer; define $Q := \check{\mathcal{S}}_{21}^{-1}\check{\mathcal{S}}_{12}^{T}$, and let κ be the number of distinct zeros of $\det Q$ in $\mathbb{C}^{+}$ with odd algebraic multiplicity. Then S has a symmetric inner completion of degree $n + \kappa$. This completion of S has minimal degree among all the symmetric completions of S.*

Proof. We apply Theorem 4 to obtain a completion of degree

$$2n - n_0 - 2\ell = n + \kappa.$$

We now prove that this completion has minimal McMillan degree. Let Σ be any symmetric completion of S. By Proposition 5, it can be written on the form

$$\Sigma = \begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix} \mathcal{S}_1 \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}$$

where L, R and $\mathcal{S}_1$ are inner, and $\mathcal{S}_1$ is a minimal degree inner completion of S. Let

$$\mathcal{S}_1 = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}.$$

The completion Σ being symmetric, we must have

$$(LS_{12})^T = S_{21}R \Leftrightarrow S_{21}^{-1}S_{12}^T = R\bar{L}.$$

From Lemma 6 we know that the degree of the unitary matrix $S_{21}^{-1}S_{12}^T$ cannot be less than κ. This yields

$$\kappa \leq \deg S_{21}^{-1}S_{12}^T = \deg R\bar{L} \leq \deg R + \deg L,$$

and finally,

$$n + \kappa \leq n + \deg R + \deg L = \deg \Sigma.$$

Extensions of higher dimension

We have seen that, if we impose the extension of the $p \times p$ matrix to be of dimension $2p \times 2p$, we cannot, in general obtain a symmetric extension which preserves the McMillan degree. Quite surprisingly, we can obtain a realization of the same McMillan degree as S if we allow for a slightly bigger extension, namely one of dimension $(2p+1) \times (2p+1)$.

Theorem 8 ***Let S be a strictly contractive symmetric $p \times p$ Schur function of degree n. Then S has a symmetric inner extension of dimension $(2p+1) \times (2p+1)$ and McMillan degree n.***

Proof. Let $\check{\mathcal{S}} = \begin{bmatrix} \check{\mathcal{S}}_{11} & \check{\mathcal{S}}_{12} \\ \check{\mathcal{S}}_{21} & S \end{bmatrix}$ **be the minimal extension S**

with $\check{\mathcal{S}}_{21}$ outer. We know, from Lemma 3, that $Q = \check{\mathcal{S}}_{21}^{-1}\check{\mathcal{S}}_{12}^{T}$ is inner and that

$$\check{\mathcal{S}} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \check{\mathcal{S}}_{11}Q & \check{\mathcal{S}}_{12} \\ \check{\mathcal{S}}_{12}^{T} & S \end{bmatrix}$$

is symmetric of degree $2n - n_0$, where n_0 is the number of zeros of $\check{S}_{21}$ on the imaginary axis. But then the matrix

$$\mathcal{S}_e := \begin{bmatrix} \check{\mathcal{S}}_{11}Q & 0 & \check{\mathcal{S}}_{12} \\ 0 & \det Q & 0 \\ \check{\mathcal{S}}_{12}^{T} & 0 & S \end{bmatrix}$$

is inner of degree $n + 2(n - n_0)$ and the matrix

$\mathcal{Q} := \begin{bmatrix} Q & 0 \\ 0 & \det Q \end{bmatrix}$ **has exactly $n - n_0$ double zeros (all the zeros of $\mathcal{Q}$ are double!). But then, in view of Lemma ??, we can obtain a reduction of degree by $2(n - n_0)$, as wanted.**

Remark 9 ***It should be stressed that the above results rely on the fact that we work over the complex field. The situation for real coefficients functions is more complicated, as shall be seen next.***

Extensions of higher dimension which preserve the degree

We have seen that, if we impose the extension of the $p \times p$ matrix to be of dimension $2p \times 2p$, we cannot, in general obtain a symmetric extension which preserves the McMillan degree. Quite surprisingly, we can obtain a realization of the same McMillan degree as S if we allow for a slightly bigger extension, namely one of dimension $(2p + 1) \times (2p + 1)$.

Theorem 10 *Let S be a strictly contractive symmetric $p \times p$ Schur function of degree n. Then S has a symmetric inner extension of dimension $(2p+1) \times (2p+1)$ and McMillan degree n.*

Proof. Let $\check{\mathcal{S}} = \begin{bmatrix} \check{S}_{11} & \check{S}_{12} \\ \check{S}_{21} & S \end{bmatrix}$ be the minimal extension S with $\check{S}_{21}$ outer. We know, from Lemma 3, that $Q = \check{S}_{21}^{-1} \check{S}_{12}^T$ is inner and that

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Remark 11 ***It should be stressed that the above results rely on the fact that we work over the complex field. The situation for real coefficients functions is more complicated.***

The real case

To avoid getting lost with signatures, we need:

Lemma 12 ***Let T be a symmetric inner function. Suppose B_1 and B_2 are degree one Blaschke factors with a zero in ω_1 and ω_2 respectively, $\omega_1 \neq \omega_2$, and that T factors as***

$$\begin{aligned} T &= B_1^T T_1 B_1 \\ T &= B_2^T T_2 B_2 \end{aligned} \qquad (2)$$

with T_1 and T_2 analytic. Then $T = B^T T_0 B$ where T_0 is analytic and B is the Least Common Right Multiple (LCRM) of B_1 and B_2.

If T has real coefficients, and $B_2 = \bar{B}_1$, then B and T_0 are real.

Proof. Let $\tilde{B}_2 := BB_1^*$ and $\tilde{B}_1 := BB_2^*$; since B is the LCRM of B_1, B_2, the functions $\tilde{B}_1$ and $\tilde{B}_2$ are left coprime. Since B has a single zero both in ω_1 and ω_2, $\tilde{B}_2$ must have a zero in ω_2 and $\tilde{B}_1$ a zero in ω_1. Using the equality $B_1B_2^* = \tilde{B}_2^*\tilde{B}_1$,

$$B_1^T T_1 = TB_1^* = B_2^T T_2 B_2 B_1^* = B_2^T T_2 \tilde{B}_1^* \tilde{B}_2 \qquad (3)$$

The term on the right-hand side is analytic, and $\tilde{B}_2$ and $\tilde{B}_1$ are left coprime. This means that there exists stable matrix functions X, Y such that

$$\tilde{B}_1^* \tilde{B}_2 Y = \tilde{B}_1^* - X. \qquad (4)$$

Multiplying the last term of (3) by Y, we get:

$$B_2^T T_2 \tilde{B}_1^* \tilde{B}_2 Y = B_2^T T_2 \tilde{B}_1^* - B_2^T T_2 X,$$

so that $B_2^T T_2 \tilde{B}_1^*$ is analytic; B_2^T is a simple Blaschke factor with a zero in ω_2, so it has full rank in ω_1. Therefore also $T_2 \tilde{B}_1^*$ is analytic.

Consider now

$$T_1 = (B_1^T)^* B_2^T T_2 \tilde{B}_1^* \tilde{B}_2 = \tilde{B}_2^T (\tilde{B}_1^*)^T T_2 \tilde{B}_1^* \tilde{B}_2$$

In view of (4) we can write:

$$\begin{aligned} & Y^T T_1 Y \\ = \; & (\tilde{B}_1^* - X)^T T_2 (\tilde{B}_1^* - X) \\ = \; & (\tilde{B}_1^*)^T T_2 \tilde{B}_1^* - X^T T_2 \tilde{B}_1^* - (\tilde{B}_1^*)^T T_2 X + X T_2 X \end{aligned}$$

which implies that $T_0 = (\tilde{B}_1^*)^T T_2 \tilde{B}_1^*$ is analytic, since all the other terms are.

If T has real coefficients and $B_2 = \bar{B}_1$, since B is the LCLM of B_1 and B_2, it is invariant under conjugation, and so it must have real coefficients, and thus so does T_0.

Lemma 13 *Let T be a real coefficients $p \times p$ stable symmetric function having a real zero ω_l of multiplicity r_l and such that all partial multiplicities of ω_l are equal to 1. Factorize T as $T = T_1 Q_l$, where $Q_l(s) = U B_{r_l,\omega_l}(s) U^*$ and U is unitary; define $Z := \ker Q_l(\omega_l)$ and let i_l be the signature of the matrix $T_1(\omega_l)_{|Z}$. Then T admits the analytic real coefficients factorization $T = B^T T_2 B$ with B inner of degree $(r_l - |i_l|)/2$ and T_2 stable.*

Proof. the idea is to use the same argument as in the complex case, that is, to find a vector u such that

$$u^T T_1(\omega_l) u = 0$$

Nevertheless, while this is always possible in the complex case, here it can only be done if the metric induced by $T_1(\omega_l)$ is indefinite.

For example

$$u^T \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} u = 0$$

has solution $u^T = [1, 1]$**, while**

$$u^T \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} u = 0$$

has only the zero solution.

the general result is thus as follows:

Minimal dimension real extension

Theorem 14 *Let $\mathcal{S}$ be a symmetric extension of a strictly contractive symmetric $p \times p$ Schur function of degree n with real coefficients; let n_c be the number of complex zeros with odd multiplicity and let $\omega_l, 1 \leq l \leq k$ be its real zeros ; let i_l be defined as in Lemma 13 for $1 \leq l \leq k$ and set*

$$n_r := \sum_{l=1}^{k} |i_l|$$

Then S has a symmetric real coefficients inner extension of dimension $(2p) \times (2p)$ and McMillan degree $n + n_c + n_r$. The degree of the extension is minimal.

PROOF. **In view of Lemma 12, we can get rid of all double complex zeros (because also the conjugate will be double and thus we can factor out the real least common multiple). As for the real zeros, put together Lemma 13 and Lemma 12.**

Degree n real extension of higher dimension

As in the complex case, we can try to increase the dimension beyond $2p$ while keeping the degree equal to n. Results by Vongpanitlerd (1970) show that there exists such an extension of dimension $2p + n$. Nevertheless it's not difficult to see that a better bound can be achieved. In fact, we can always write an extension $\mathcal{S}$ in Smith-McMillan form as

$$\mathcal{S} = \pi^T \delta^{-1} \sigma_{sm} \pi$$

where δ and σ_{sm} are diagonal polynomial and π is a real unimodular polynomial matrix. Since $\mathcal{S}$ is inner, its zeros are antistable and thus $S_{zeros} := \sigma_{sm} \sigma_{sm}^{-*}$ is inner. Since the reduction to the Smith-McMillan form of $\mathcal{S}$ does not change its

signature, the previous results show that, if we take

$$\mathcal{S}_e := \begin{bmatrix} \mathcal{S} & \\ & -S_{zeros} \end{bmatrix}$$

we easily see that $\mathcal{S}_e$ has only double zeros and the residual matrix of the real zeros has zero signature. Thus, using Lemma 12, it can be reduced to an extension of degree n.

In fact, this reduction might still not be minimal. For a minimal one we have the following:

Theorem 15 *Let $\mathcal{S}_{sm}$ be the Smith-McMillan form of T and factor it as*

$$\mathcal{S}_{sm} = \delta^{-1}\sigma_c\sigma_{rd}\tau_{rs}$$

where all the matrices are diagonal, δ is the denominator matrix, τ_c contains all the complex zeros, τ_{rd} contains the highest even number of real zeros of geometric multiplicities greater than one and τ_{rs} has only real zeros with partial multiplicities equal to 1. Let r be the number of non constant diagonal entries of τ_{rs} Then there exists an extension of dimension $2p + r$ and degree n.

Proof. Set

$$S_{mz} := \tau_{rs}\tau_{rs}^{-*} \begin{bmatrix} I_{r-1} & \\ & \det \frac{\sigma_c}{\sigma_c^{-*}} \end{bmatrix}$$

The zeros of σ_c are all complex and will not change the signature. Then

$$\mathcal{S}_e := \begin{bmatrix} \mathcal{S} & \\ & -S_{mz} \end{bmatrix}$$

has the wanted dimension, double zeros with the right signature and thus it can be reduced to degree n.

In conclusion...

Thank you for your attention!

Inner extension

From Darlington synthesis it's well known that, if S is a Schur function of degree n, then there exists an inner completion of the same degree. In the case a function is wide inner, its completion can be easily computed using state space formulas (see e.g. Fuhrmann 1995); nevertheless, a frequency domain expression which makes use of the information about the entries seem to be lacking. The following lemma provides one.

Theorem 16 *Let S be a $p \times p$ Schur function strictly contractive at some point of $i\mathbb{R}$, and such that $I + S$ is invertible in H^∞. Let S_{21} be a spectral factor of $I_p - SS^*$. Every inner completion $\mathcal{S}$ of $[S_{21}\ S]$ can be written as:*

$$\mathcal{S} = \begin{bmatrix} M - MS_{21}^*(I + S^*)^{-1}S_{21} & -MS_{21}^*(I + S^*)^{-1}(I + S) \\ S_{21} & S \end{bmatrix} \tag{5}$$

where M is a left inner factor of $(I + S)^{-1}S_{21}$, i.e.

$$MS_{21}^*(I + S^*)^{-1} \tag{6}$$

is stable. The extension $\mathcal{S}$ has same degree as $[S_{21}\ S]$ if and only if M has minimal degree (i.e. it is the DSS factor).

PROOF. It is easily verified that (5) provides an inner extension of $[S_{21}\ S]$. Conversely, to see that every inner extension can be written in this form, let

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix}$$

be an inner completion of $[S_{21}\ S]$ and put

$$M = S_{11} - S_{12}(I_p + S)^{-1} S_{21}. \tag{7}$$

Using the fact that $\mathcal{S}$ is inner, i.e.

$$\begin{aligned} S_{11}S_{11}^* + S_{12}S_{12}^* &= I_p \\ S_{11}S_{21}^* + S_{12}S^* &= 0 \\ S_{21}S_{21}^* + SS^* &= I_p \end{aligned} \tag{8}$$

and (7), we can show that

$$\begin{aligned} MM^* &= S_{11}S_{11}^* - S_{11}S_{21}^*(I_p + S^*)^{-1}S_{12}^* - S_{12}(I_p + S)^{-1}S_{21}S_{11}^* \\ &\quad + S_{12}(I_p + S)^{-1}(I - SS^*)(I_p + S^*)^{-1}S_{12}^* = I \end{aligned}$$

i.e. M is inner and

$$S_{12} = -MS_{21}^*(I + S^*)^{-1}(I + S). \tag{9}$$

Thus

$$\begin{aligned} S_{11} &= M + S_{12}(I_p + S)^{-1}S_{21} && (10) \\ & \ M - MS_{21}^*(I + S^*)^{-1}S_{21} && (11) \end{aligned}$$

Notice that $-S_{12}(I+S)^{-1}$ is stable whereas $S_{21}^{*}(I+S^{*})^{-1}$ is antistable, so that M is a left inner factor of $S_{12}(I+S)^{-1}$.

Let n' be the degree of $[S_{21}\ S]$. It can then be shown that $\mathcal{S}$ has degree n' if and only if M and $S_{12}(I+S)^{-1}$ are left coprime.

Back to (1).

A few definitions

To fix ideas, we work with functions analytic in $\mathbb{C}^+$

- **We define for a rational function** $R(s)$ **the function** R^* **as:**

$$R^*(s) := R^T(-s) \tag{12}$$

- **We say that a function analytic in** $\mathbb{C}^+$ **is Schur if**

$$R(i\omega)R^*(i\omega) \leq I \qquad \omega \in \mathbb{R} \cup \infty \tag{13}$$

- **A Schur function is inner if**

$$R(i\omega)R^*(i\omega) = I \qquad \omega \in \mathbb{R} \tag{14}$$