INTERCONNECTION AND DECOMPOSITION PROBLEMS WITH nD BEHAVIORS

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OUTLINE

- Introduction : Motivation and aim.
- Modules and behaviors.
- Regular interconnection, controllable and autonomous
- Translating system problems to algebraic problems.
- A useful theorem.
- Controllable-autonomous decomposition with finite dimensional intersection.
- Almost regular interconnection.

INTRODUCTION

The behavioral approach is an elegant framework for the analysis and design of multidimensional systems.

Regular Interconnection problem:

- **Given a plant behavior $\mathfrak{B}$ and a control objective $\mathfrak{B}_d$ find a controller behavior $\mathfrak{B}_c$ such that the interconnection of $\mathfrak{B}$ and $\mathfrak{B}_c$ gives $\mathfrak{B}_d$.**
- **If the restrictions of $\mathfrak{B}_c$ are independent of the restrictions of the plant then the interconnection is *regular* (as in a feedback loop).**

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Autonomous-controllable Decomposition problem:

• When is it possible to write a system as the direct sum of its controllable part and an autonomous part?

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1D case:

- **Regular Interconnection problem studied by J.C. Willems.**
- **Auton-control decomposition problem studied by Kalman.**

nD case:

- **Regular Intercon. problem studied by P. Rocha, E.Zerz, J. Wood...**
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Very strong conditions needed for solving the problems in nD systems!

When does there exists $\mathfrak{B}_c$ such that the regular interconnection of $\mathfrak{B}$ and $\mathfrak{B}^c$ gives "almost" $\mathfrak{B}^d$?

When can we decompose the system into the "almost" direct sum of the controllable part and the autonomous part?

What does "almost" mean?

MULTIDIMENSIONAL SYSTEMS

The system

$$\boldsymbol{\Sigma = (\mathcal{A}, q, \mathfrak{B})}$$

$\mathfrak{B}$ = **Solution set of a system of linear constant-coefficient PDE's.**

$\mathcal{A} = \mathfrak{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$ **or** $D'(\mathbb{R}^n, \mathbb{R})$ **(for discrete case** $\mathcal{A} = \mathbb{R}^{\mathbb{N}^n}$**)**

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$\mathfrak{D} := \mathbb{R}[\xi_1, ..., \xi_n]$ **the polynomial ring.**

We can find $R \in \mathfrak{D}^{\bullet \times q}$ **such that** $\boxed{R(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})w = 0}$

$\mathfrak{B} = \{w \in \mathcal{A}^q \mid$ **the equation holds**$\} =$ **ker**(R)

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- **For discrete case take the shift operator** σ_i **instead of** $\frac{\partial}{\partial x_i}$**.**

MODULES AND BEHAVIORS

$\mathfrak{B}^{\perp}$:= $<R>$ = module of $\mathfrak{D}^q$ generated by the rows of R

- Oberst showed that $\mathfrak{B} \overset{1:1}{\longleftrightarrow} \mathfrak{B}^{\perp}$

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- $\mathfrak{B}_1 \subseteq \mathfrak{B}_2 = \mathfrak{B}_1^{\perp} \supseteq \mathfrak{B}_2^{\perp}$

- $(\mathfrak{B}_1 + \mathfrak{B}_2)^{\perp} = \mathfrak{B}_1^{\perp} \cap \mathfrak{B}_2^{\perp}$

- $(\mathfrak{B}_1 \cap \mathfrak{B}_2)^{\perp} = \mathfrak{B}_1^{\perp} + \mathfrak{B}_2^{\perp}$

- $\mathfrak{B}$ has finite dimension over $\mathbb{R}$ $\iff$ $\mathfrak{B}^{\perp} \subseteq \mathfrak{D}^q$ has finite codimension (i.e. $\mathfrak{D}^q/\mathfrak{B}^{\perp}$ has finite dimension).

INTERCONNECTION

Interconnection of systems is the basis for control in the behavioral framework and are descrived by interconnections of the corresponding behaviors. Thus

$$\mathfrak{B}_1 = \text{ker}(R_1),\ \mathfrak{B}_2 = \text{ker}(R_2) \ \Rightarrow \mathfrak{B}_1 \cap \mathfrak{B}_2 = \text{ker}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

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$$\text{rank}(R_1) + \text{rank}(R_2) = \text{rank}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \text{ or if } \boxed{\mathfrak{B}_1^{\perp} \cap \mathfrak{B}_2^{\perp} = 0}$$

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Expresses the idea of ”restricting what is not yet restricted”.

- **Given $\mathfrak{B}$, if there exists $\mathfrak{B}_c$ s.t. $\mathfrak{B}_d = \mathfrak{B} \cap \mathfrak{B}_c$, then $\mathfrak{B}_d$ is said to be implementable from $\mathfrak{B}$. And it is called regular implementable if the interconnection is regular.**

CONTROLLABILITY

- A $\mathfrak{B}$ is said to be controllable if for all w_1,$w_2 \in \mathfrak{B}$ and all sets U_1 ,$U_2 \subset \mathbb{R}^n$ with disjoint closure, there exist a $w \in \mathfrak{B}$ such that $w \mid_{U_1} = w_1 \mid_{U_1}$ and $w \mid_{U_2} = w_2 \mid_{U_2}$.
- Let N be a $\mathfrak{D}$-module. Then $n \in N$ is called a torsion element if there exists a nonzero $d \in \mathfrak{D}$ such that $dn = 0$.

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- A $\mathfrak{B} \subseteq \mathcal{A}$ is said to be strongly controllable if it is a direct summand of $\mathcal{A}$.
- A free module is a module having a free basis

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CONTROLLABILITY

- A $\mathfrak{B}$ is said to be **controllable** if for all w_1,$w_2 \in \mathfrak{B}$ and all sets U_1 ,$U_2 \subset \mathbb{R}^n$ with disjoint closure, there exist a $w \in \mathfrak{B}$ such that $w \mid_{U_1} = w_1 \mid_{U_1}$ and $w \mid_{U_2} = w_2 \mid_{U_2}$.
- Let N be a $\mathfrak{D}$-module. Then $n \in N$ is called a **torsion** element if there exists a nonzero $d \in \mathfrak{D}$ such that $dn = 0$.

> $\mathfrak{B}$ is controllable iff $\mathfrak{D}^q/\mathfrak{B}^\perp$ is torsion free

- A $\mathfrak{B} \subseteq \mathcal{A}$ is said to be **strongly controllable** if it is a direct summand of $\mathcal{A}$.
- A **free** module is a module having a free basis

> A $\mathfrak{B} \subseteq \mathcal{A}$ is said to be strongly controllable iff $\mathfrak{D}^q/\mathfrak{B}^\perp$ is free

- $\mathfrak{B}$ is **autonomous** if $\mathfrak{D}^q/\mathfrak{B}^\perp$ is a **torsion module**

"Almost" regular implementability problem

"Almost" regular implementability problem:

Given $\mathfrak{B}$ and a desired $\mathfrak{B}_d \subseteq \mathfrak{B}$ when does there exists $\mathfrak{B}_c$ s.t. $\mathfrak{B}_d \subseteq \mathfrak{B} \cap \mathfrak{B}_c$ finite codimension with regular interconnection?

Equivalent to the algebraic problem:

Given are $N_1 \subseteq N \subseteq \mathfrak{D}^q$ when does there exists $N_2 \subseteq N \subseteq \mathfrak{D}^q$ s.t. $N_1 + N_2 \subseteq N$ finite codimension and $N_1 \cap N_2 = 0$?

- Note that $N_1 = \mathfrak{B}^\perp, \ N = \mathfrak{B}_d^\perp, \ N_2 = \mathfrak{B}_c^\perp$

"Almost" direct summand controllable-autonomous decomposition

"Almost" direct summand control-aut decomposition problem:

Given $\mathfrak{B}$, is it possible to decompose it as

$\mathfrak{B} = \mathfrak{B}_{contr} + \mathfrak{B}_{aut}$ **and** $\mathfrak{B}_{contr} \cap \mathfrak{B}_{aut}$ **finite dimension**

Equivalent to the algebraic problem:

Given a $\mathfrak{D}$-module M. Let M_t be its torsion part, does there exists a $\mathfrak{D}$-module A such that

$M_t + A \subseteq M$ **with finite codimension and** $M_t \cap A = 0$

A useful theorem

- **From now on take $\mathfrak{D} = \mathbb{R}[\xi_1, \xi_2]$.**

Theorem: Let M be a finitely generated $\mathfrak{D}$-module with no torsion. Then there exists a free $\mathfrak{D}$-module M^+ such that $M \subseteq M^+$ and M^+/M has finite dimension.

A useful theorem

- **From now on take $\mathfrak{D} = \mathbb{R}[\xi_1, \xi_2]$.**

Theorem: Let M be a finitely generated $\mathfrak{D}$-module with no torsion. Then there exists a free $\mathfrak{D}$-module M^+ such that $M \subseteq M^+$ and M^+/M has finite dimension.

Corollary: Given a controllable $2D$ behavior $\mathfrak{B} \subseteq \mathcal{A}^q$ there exists a strongly controllable sub-behavior $\mathfrak{B}' \subseteq \mathfrak{B}$

Solution to the "almost" regular implementability problem

Given $\mathfrak{B}$ and a desired $\mathfrak{B}_d \subseteq \mathfrak{B}$ there exists $\mathfrak{B}_c$ s.t. $\mathfrak{B}_d \subseteq \mathfrak{B} \cap \mathfrak{B}_c$ finite codimension with regular interconnection

$$\Updownarrow$$

Given are $N_1 \subseteq N \subseteq \mathfrak{D}^q$ there exists $N_2 \subseteq N \subseteq \mathfrak{D}^q$ s.t. $N_1 + N_2 \subseteq N$ finite codimension and $N_1 \cap N_2 = 0$

$$\Updownarrow$$

Given are $N_1 \subseteq N \subseteq \mathfrak{D}^q$, take $N_2 \subseteq N_2^+$ finite codimension and $N \subseteq N^+$ finite codimension then N_1^+ is direct summand of N^+

- Remember that $N_1 = \mathfrak{B}^\perp$, $N = \mathfrak{B}_d^\perp$
- Direct summands can be computed!!

Solution to "almost" direct summand control-aut decomposition

Given $\mathfrak{B}$, is it possible to decompose it as
$\mathfrak{B} = \mathfrak{B}_{contr} + \mathfrak{B}_{aut}$ and $\mathfrak{B}_{contr} \cap \mathfrak{B}_{aut}$ finite dimension

$\Updownarrow$

Given a $\mathfrak{D}$-module M. Let M_t be its torsion part, there exists a $\mathfrak{D}$-module A such that $M_t + A \subseteq M$ with finite codimension and $M_t \cap A = 0$

Solution: Yes, it is possible of n=2.

Example: Take $n = 3$ i.e. $\mathfrak{D} = \mathbb{R}[\xi_1, \xi_2, \xi_3]$ then

$\mathfrak{B} = ker \begin{pmatrix} \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} & -\frac{\partial^2}{\partial x_1^2} \end{pmatrix}$ is not an "almost" direct summand of its controllable part and an autonomous part.