MODELING/IDENTIFICATION BY RECIPROCAL PROCESSES

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MOTIVATION: MODELING (IDENTIFICATION) OF IMAGES

Gibbs-Markov models for random fields (Geman & Geman, Chellappa & Kashyap, Balram & Moura...) lead to difficult estimation problems: simulated annealing, MCMC, etc..

Reciprocal processes are G-M random fields well studied in 1-D, admit *descriptor type* representation (Krener, Levy, Frezza ,...) natural **non-causal** extension of linear state space models

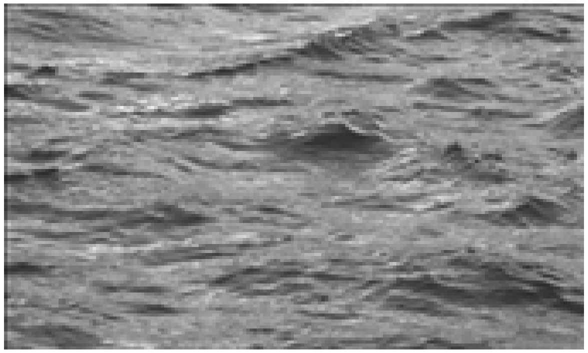
Identification (subspace) using linear local models structure could be much easier.

TEXTURES: WHAT ARE THEY

Image on a (finite) 2-D lattice $\equiv$ (stochastic) random field

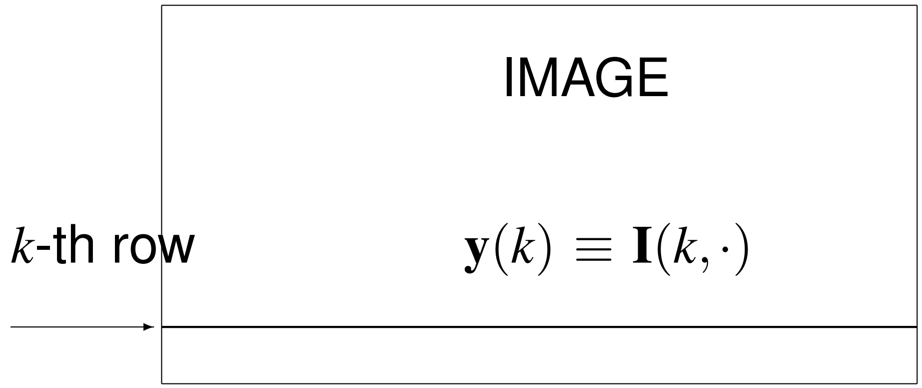
$$\{\mathbf{I}(k,h)\,;\, k = 1,\dots,N,\ h = 1,\dots,m\}$$

Texture : $\equiv$ a *(spatially) stationary* random field e.g. River sequence



1-D MODELING OF TEXTURES

$N \times m$ pixels:



$\{\mathbf{y}(k), k = 1, 2, \ldots, N\}$ m-dimensional stochastic process (assume zero mean)

MODEL: to describe the correlation structure of $\{\ldots \mathbf{y}(k-1), \mathbf{y}(k), \mathbf{y}(k+1), \ldots\}$

STATIONARY PROCESSES ON A FINITE INTERVAL

Write $\mathbf{y} := \{\mathbf{y}(k), k = 1, 2, \ldots, N\}$ as a column vector with N (m-dimensional) components. We shall say that $\mathbf{y}$ is *stationary* if

$$\mathbf{R} := \mathbb{E}\mathbf{y}\mathbf{y}^\top = \begin{bmatrix} R(0) & R(1) & \ldots & R(N-1) \\ R(1)^\top & R(0) & R(1) & \ldots \\ \ldots & & & \ldots \\ R(N-1)^\top & \ldots & R(1)^\top & R(0) \end{bmatrix}$$

(block-Toeplitz structure).

$\mathbf{y}$ of **full rank** (or *minimal*) if covariance matrix $\mathbf{R} := \mathbb{E}\mathbf{y}\mathbf{y}^\top$ is positive definite.

PERIODIC STATIONARY PROCESSES

Assume $\mathbf{y}$ is a periodic process of period N: $\mathbf{y}(k+\nu N) := \mathbf{y}(k)$ for arbitrary $\nu \in \mathbb{Z}$. Make it a process on the *discrete group* $\mathbb{Z}_N \equiv \{1, 2, \ldots, N\}$ with arithmetics mod N. Then for $\tau = 0, 1, \ldots, N/2$ (assume N even)

$$R(N/2+\tau) = \mathbb{E}\mathbf{y}(t+\tau+N/2)\mathbf{y}(t+N)^\top = R(\tau - N/2) = R(N/2-\tau)^\top$$

For $\tau = N/2 - 1$

$$R(N/2+\tau) = R(N/2-\tau)^\top \Rightarrow R(1) = R(N/2+N/2-1) = R(N-1)^\top$$

(Also $R(N) = R(0)$ etc..)

Fact: A periodic stationary process of period N must have a **block-circulant** covariance matrix.

(BANDED) BLOCK-CIRCULANT MATRICES

$$
\mathbf{C}_N = \begin{bmatrix}
R(0) & R(1) & \dots & R(p) & 0 & \dots & 0 & R(p)^\top & \dots & R(1)^\top \\
R(1)^\top & R(0) & R(1) & \dots & R(p) & 0 & \dots & 0 & \ddots & \vdots \\
\vdots & & \ddots & & & \ddots & & \vdots & & R(p)^\top \\
R(p)^\top & \dots & R(1)^\top & R(0) & R(1) & \dots & R(p) & 0 & & 0 \\
0 & R(p)^\top & & \dots & R(0) & \dots & & R(p) & & \vdots \\
\vdots & & & \dots & & \dots & & & & 0 \\
0 & & & \dots & & \dots & & & & R(p) \\
R(p) & 0 & \ddots & & & & & & & \vdots \\
\vdots & \ddots & & \ddots & & & & & \ddots & R(1) \\
R(1) & \dots & R(p) & 0 & \dots & 0 & R(p)^\top & \dots & R(1)^\top & R(0)
\end{bmatrix}
$$

Notation (for N blocks)

$$
\mathbf{C}_N = \mathrm{Circ}\{R(0), R(1), \dots, R(p), 0, \dots, 0, R(p)^\top, \dots, R(1)^\top\}\,.
$$

SYMBOL OF A (BANDED) BLOCK-CIRCULANT

Discrete Fourier Transform of the finite sequence of N blocks

$$\{R(0), R(1), \ldots, R(p), 0, \underset{\substack{\uparrow \\ N/2}}{\ldots}, 0, R(p)^\top, \ldots, R(1)^\top\}; \qquad R(\tau)^\top = R(-\tau)$$

$$\Phi(z_k) := \sum_{\tau=-N/2}^{(N/2)} R(\tau)\, z_k^{-\tau}, \qquad R(\tau) = \frac{2}{N} \sum_{k=-N/2}^{(N/2)} \Phi(z_k)\, z_k^{\tau} \qquad ((N/2) := N/2 - 1)$$

$z \equiv z_k$ runs on the *discrete unit circle* $\mathbb{C}_N$; i.e. $z_k = \rho^k,\ k = 0, 1, 2, \ldots, N-1$, where $\rho = e^{i2\pi/N}$ = primitive N-th root of unity.

More usual (but less convenient):

$$R(0)z^0 + R(1)z^{-1} + \ldots + R(p)z^{-p} + 0 \ldots + 0 + R(p)^\top z^{-N+p} + \ldots + R(1)^\top z^{-N+1}$$

PERIODIC EXTENSION OF A STAT. PROCESS

Can “extend ” $\mathbf{y}$ to $[N,\ 2N-1]$ and make it a periodic process of period $2N$: $\mathbf{y}(k+2\nu N) := \mathbf{y}(k)$ Augment the covariance to $2Nm \times 2mN$ so as to make it block-circulant.

$$R(N+\tau) := R(N-\tau)^\top, \qquad \tau = 0, 1, \dots N-1$$

Use only known data !! e.g. for $\tau = N-1$, $R(2N-1) = R(1)^\top$ etc.

$$\mathbf{R} = \begin{bmatrix} R(0) & R(1) & \dots & R(2N-1) \\ R(2N-1) & R(0) & R(1) & \dots \\ \dots & & & \dots \\ R(1) & \dots & R(2N-1) & R(0) \end{bmatrix} = \begin{bmatrix} R(0) & R(1) & \dots & R(1)^\top \\ R(1)^\top & R(0) & R(1) & \dots \\ \dots & & \ddots & \dots \\ R(1) & \dots & R(1)^\top & R(0) \end{bmatrix}$$

We don’t NEED samples of the process on the half-period $[N,\ 2N-1]$. Possible to extend the original $\mathbf{R}$ to a smaller interval to make it block-circulant (Dembo, Mallows, Shepp 1989).

SPECTRUM OF A STATIONARY PERIODIC PROCESSES

Spectrum = symbol of the block-circulant covariance matrix $\mathbf{R}$

$$\Phi(z) := \sum_{\tau=-N}^{(N)} R(\tau) z^{-\tau}$$

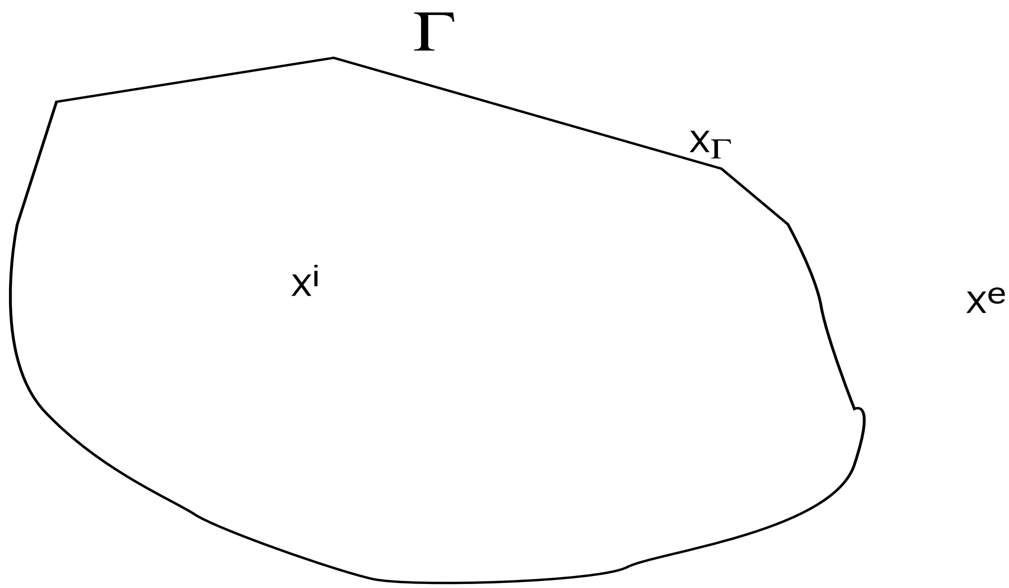
The variable $z \equiv e^{i\Delta k}$ runs on the *discrete unit circle* $\mathbb{C}_{2N}$ sampled with angular width $\Delta = \frac{2\pi}{2N}$, starting at $\theta_0 = -\pi$ and travelled counterclockwise up to the endpoint $\theta = +\pi$ (excluded).

$$R(\tau) := \frac{1}{N} \sum_{k=-N}^{(N)} \Phi(\rho^k) \rho^{k\tau} \qquad \rho := e^{i\Delta}$$

"STATE SPACE MODELS" FOR PROCESSES ON A FINITE INTERVAL ($\mathbb{Z}_N$)

Q: What is a natural notion of state on a finite interval?

MARKOV PROPERTY FOR RANDOM FIELDS



INTERIOR-AND-EXTERIOR-ARE-CONDITIONALLY-INDEPENDENT-GIVEN-VALUES-AT THE-BOUNDARY

1-D RECIPROCAL PROCESSES

A n-dimensional process $\mathbf{x} := \{\mathbf{x}(k), k \in \mathbb{Z}_N\}$ is *reciprocal* if for all $k \in (k_0, k_1)$ and h in the complementary interval $(k_0, k_1)^c$ ($\mod N$) $\mathbf{x}(k)$ and $\mathbf{x}(h)$ **conditionally uncorrelated given the boundary values** $\mathbf{x}(k_0)$ and $\mathbf{x}(k_1)$.

$$\mathbf{x}_{(k_0,k_1)} \perp \mathbf{x}_{(k_0,k_1)^c} \mid \{\mathbf{x}(k_0), \mathbf{x}(k_1)\}$$

Fact: The best linear estimate of $\mathbf{x}(k)$ given all other $\mathbf{x}(h), h \neq k$ is

$$\mathbb{E}\{\mathbf{x}(k) \mid \mathbf{x}(h), h \neq k\} = F_+\mathbf{x}(k+1) + F_-\mathbf{x}(k-1)$$

$\mathbf{x}$ is said to be of *full rank* (or *minimal*) if the $(nN \times nN)$ covariance matrix $\Sigma := \mathbb{E}\mathbf{x}\mathbf{x}^\top$ is positive definite.

STATIONARY RECIPROCAL PROCESSES (1)

THEOREM 1 *Every stationary reciprocal process* **of full rank** *on* $\mathbb{Z}_N$ *can be represented by a three terms recursion of the following form*

$$M\mathbf{x}(k) = F^\top \mathbf{x}(k-1) + F\mathbf{x}(k+1) + \mathbf{e}(k) \qquad (*)$$

where M, F *are constant matrices,* M *is symmetric and positive definite, and* $\mathbf{e}$ *is a locally correlated process, i.e*

$$\mathbb{E}\,\mathbf{e}(k)\mathbf{e}(h)^\top = 0 \qquad |k-h| > 1$$

such that

$$\mathbb{E}\,\mathbf{x}(k)\mathbf{e}(k)^\top = I \qquad \mathbb{E}\,\mathbf{x}(k)\mathbf{e}(h)^\top = 0 \qquad k \neq h$$

N.B. $\mathbf{e}(k) = \mathbf{w}(k) + B\mathbf{w}(k-1)$ for some white noise $\mathbf{w}$.

DYNAMICAL EQUATION FOR RECIPROCAL PROCESSES

Extend $\mathbf{x}$ (periodically) to $\mathbb{Z}_N$ so $\mathbf{x}(k+N) = \mathbf{x}(k)$. Dynamical model (*) has *cyclic boundary conditions* $\mathbf{x}(N) = \mathbf{x}(0)$.

$$\begin{bmatrix} M, & -F & 0 & \dots & 0 & -F^\top \\ -F^\top & M & -F & \dots & 0 & 0 \\ 0 & -F^\top & M & -F & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & -F^\top & M & -F \\ -F & \dots & \dots & \dots & -F^\top & M \end{bmatrix} \begin{bmatrix} \mathbf{x}(1) \\ \mathbf{x}(2) \\ \vdots \\ \mathbf{x}(N-1) \\ \mathbf{x}(N) \end{bmatrix} = \begin{bmatrix} \mathbf{e}(1) \\ \mathbf{e}(2) \\ \vdots \\ \mathbf{e}(N-1) \\ \mathbf{e}(N) \end{bmatrix}$$

Denote, $\Lambda := \mathrm{Circ}\left[M, -F, 0, \dots, 0, -F^\top\right]$

FULL-RANK STATIONARY RECIPROCAL PROCESSES

$$\Lambda \mathbf{x} = \mathbf{e} \qquad \Rightarrow \qquad \Lambda\, \mathbb{E}\,\mathbf{x}\mathbf{x}^\top = \mathbb{E}\,\mathbf{e}\mathbf{x}^\top = I \qquad \Rightarrow \qquad \mathbb{E}\,\mathbf{x}\mathbf{x}^\top = \Lambda^{-1}$$

FACT: *The covariance of a full-rank reciprocal stationary process* $\mathbf{x}$ *on* $\mathbb{Z}_N$ *must be the inverse of a block-tridiagonal circulant matrix.*

$$\boldsymbol{\Sigma} := \mathbb{E}\,\mathbf{x}\mathbf{x}^\top = \Lambda^{-1} = \mathrm{Circ}\left[M, -F, 0, \dots, 0, -F^\top\right]^{-1}$$

For Gibbs-Markov models $\Lambda = \boldsymbol{\Sigma}^{-1}$ **Potential Matrix.**

MODELS OF NON MINIMAL R–PROCESSES

For *non-minimal* $\mathbf{x}$ tridiagonal structure no longer holds! Instead $\mathbf{x}(k) \simeq \mathbf{z}(k) \in \mathbb{R}^q, q < n$ of full rank, but smaller dimension, where

$$M_0 \mathbf{z}(k) = - \sum_{|j| \leq \nu,\ j \neq 0} M_j \mathbf{z}(k-j) + \mathbf{e_z}(k)$$

$\mathbf{e_z}$ is the **conjugate process** of $\mathbf{z}$: such that

$$\mathbb{E}\,\mathbf{z}(k)\mathbf{e_z}(k)^\top = I \qquad \mathbb{E}\,\mathbf{z}(k)\mathbf{e_z}(h)^\top = 0 \qquad k \neq h$$

$\mathbf{e_z}$ is a locally correlated process,

$$\mathbb{E}\,\mathbf{e_z}(k)\mathbf{e_z}(h)^\top = 0 \qquad |k-h| > \nu$$

i.e. $\mathbf{e_z}(k) = \mathbf{w}(k) + B_1 \mathbf{w}(k-1) + \ldots + B_\nu \mathbf{w}(k-\nu)$ for some white noise $\mathbf{w}$.

MODELS OF NON MINIMAL R–PROCESSES (2)

$$M_0\mathbf{z}(k) = -\sum_{|j|\leq\nu,\ j\neq 0} M_j\,\mathbf{z}(k-j) + \mathbf{e_z}(k)$$

Local interaction with closest ν neighbors !! (in fact each scalar component has its own $\nu_i\,,\ i = 1,2,\ldots\nu_q$)

Observability indices!!. No need to *assume* an *a priori* nearest neighbor structure. $\nu_i\,,\ i = 1,2,\ldots\nu_q$ should be (and can be) estimated from data !

DYNAMICAL EQUATION FOR NON FULL RANK R–PROCESSES

Dynamical model (*) can be written

$$\begin{bmatrix} M_0 & M_1 & \dots & M_\nu & 0 & (*)^\top \\ M_1^\top & M_0 & M_1 & \dots & M_\nu & 0 \\ \vdots & M_1^\top & M_0 & M_1 & \dots & M_\nu \\ M_\nu^\top & \dots & \ddots & \ddots & \ddots & \\ 0 & M_\nu^\top & \dots & M_1^\top & M_0 & M_1 \\ (*) & 0 & M_\nu^\top & \dots & M_1^\top & M_0 \end{bmatrix} \begin{bmatrix} \mathbf{z}(1) \\ \mathbf{z}(2) \\ \vdots \\ \mathbf{z}(N-1) \\ \mathbf{z}(N) \end{bmatrix} = \begin{bmatrix} \mathbf{e_z}(1) \\ \mathbf{e_z}(2) \\ \vdots \\ \mathbf{e_z}(N-1) \\ \mathbf{e_z}(N) \end{bmatrix}$$

Denote, $\Lambda := \mathrm{Circ}\left[M_0, M_1, \dots, M_\nu, 0, \dots, 0, M_\nu^\top, M_{\nu-1}^\top, \dots, M_1^\top\right]$
Banded block-circulant.

RECIPROCAL STATE MODELS

Definition: *An m-dimensional stationary process $\mathbf{y} := \{\mathbf{y}(k), k \in \mathbb{Z}_N\}$ admits a reciprocal realization, if there is an n-dimensional reciprocal stationary process $\mathbf{x}$ such that*

$$\mathbf{y}(k) = C\mathbf{x}(k) \qquad k \in \mathbb{Z}_N$$

for a suitable constant matrix C.

Reciprocal realizations are of the form

$$\begin{aligned} M\mathbf{x}(k) &= F^\top \mathbf{x}(k-1) + F\mathbf{x}(k+1) + \mathbf{e}(k) \\ \mathbf{y}(k) &= C\mathbf{x}(k) \end{aligned}$$

Problem: Does any stationary process on $\mathbb{Z}_N$ admit reciprocal realizations?.

RECIPROCAL STOCHASTIC REALIZATION

State covariance $\Sigma(\tau) := \mathbb{E}\,\mathbf{x}(k+\tau)\mathbf{x}(k)^\top$ satisfies

$$M\Sigma(k) = F^\top\Sigma(k-1) + F\Sigma(k+1) + I\delta(k)$$

Problem: compute the parameters of a (minimal) realization, (C, M, F) from the output covariance data $\mathbf{R} \equiv \{R(k), k = 0, 1, \ldots, N-1\}$.

$$R(k) = C\Sigma(k)C^\top$$

Frequency domain version (for full rank)

$$\Phi(z) \overset{?}{\equiv} C(F^\top z^{-1} + Fz + M)^{-1}C^\top, \qquad z \in \mathbb{C}_N$$

IDENTIFICATION OF RECIPROCAL MODELS(1)

In our example state vector will generally have *smaller dimension than the output*: matrix C with $n < m$ (independent) columns $\mathbf{y}(k) = C\mathbf{x}(k) \quad \Rightarrow$

$$\mathbf{Y}_k = \mathrm{span}\{\mathbf{y}_i(k)\,;\, i = 1, 2, \ldots, m\} = \mathrm{span}\{\mathbf{x}_i(k)\,;\, i = 1, 2, \ldots, n\} = \mathbf{X}_k$$

the process $\mathbf{y}$ *is itself reciprocal* but in general a *singular* reciprocal process.

(Of course this will not be true in general)

IDENTIFICATION OF RECIPROCAL MODELS(2)

Estimate C: Compute the (numerical) range space of the first block row of the output covariance matrix $\mathbf{R}$, by SVD

$$\begin{bmatrix} R(0) & R(1) & R(2) & \dots \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}$$

$\Delta_2 \simeq 0$ "noise" in the data. The "denoised" range space is (approximated by) that of the matrix U_1. In this way $C = U_1$.

For any left inverse $C^\sharp$ of C, in particular $C^\sharp = U_1^\top$, form $\mathbf{x}(k) := C^\sharp \mathbf{y}(k)$: **reciprocal** with covariance

$$\boldsymbol{\Sigma} = \mathrm{diag}\{U_1^\top, \dots, U_1^\top\} \mathbf{R} \, \mathrm{diag}\{U_1, \dots, U_1\}$$

IDENTIFICATION OF R. PROCESSES (2)

PROBLEM: *Estimate the parameters* (M, F) *of a descriptor model (*) of an observed* **full-rank** *reciprocal process* $\mathbf{x}$*.*

Maximum likelihood estimation (under the assumption of a Gaussian distribution for $\mathbf{x}$),

$$p_{(M,F)}(x) = \frac{1}{\sqrt{(2\pi)^N \det\left(\Lambda^{-1}\right)}} \exp\left(-\frac{1}{2} x^\top \Lambda x\right),$$

Λ parameterized by M and F as $\Lambda := \mathrm{Circ}\left[M, -F, 0, \ldots, 0, -F^\top\right]$

Assume that T independent sample images of the same texture $\mathbf{x}$ are available $\underline{x} := \left(x^{(1)}, .., x^{(T)}\right)$

The exact log-likelihood

$$L(M,F) = \log\det(\Lambda) - \frac{1}{T}\text{Trace}\left\{\sum_{t=1}^{T}\left[x^{(t)}\right]^{\top}\Lambda x^{(t)}\right\}$$

$$= \log\det(\Lambda) - \text{Trace}\{MT_0(\underline{x})\} - \text{Trace}\{FT_1(\underline{x})\}$$

where T_0 and T_1 are *sufficient statistics* given by:

$$T_0(\underline{x}) = \frac{1}{T}\sum_{t=1}^{T}\{\sum_{k=0}^{N} x^{(t)}(k)\left[x^{(t)}(k)\right]^{\top}\}$$

$$T_1(\underline{x}) = \frac{2}{T}\sum_{t=1}^{T}\{\frac{1}{N}\sum_{j=1}^{N} x^{(t)}(k)\left[x^{(t)}(k-1)\right]^{\top}\} + \frac{2}{T}\sum_{t=1}^{T} x^{(t)}(0)\left[x^{(t)}(N)\right]^{\top}$$

From exponential class theory:

THEOREM 2 *The statistics T_0 and T_1 are Maximum Likelihood estimators for their expected values,*

$$\frac{1}{N}T_0 = \hat{\Sigma}(0) = \text{M.L. Estimator of } \Sigma(0) = \mathbb{E}\,\mathbf{x}(k)\mathbf{x}(k)^\top$$

$$\frac{1}{N}T_1 = \hat{\Sigma}(1) = \text{M.L. Estimator of } \Sigma(1) = \mathbb{E}\,\mathbf{x}(k+1)\mathbf{x}(k)^\top$$

THE COVARIANCE SELECTION PROBLEM

Recall basic constraint on the parameters (Σ uniquely determined by (M, F) !)

$$\mathbf{\Sigma}^{-1} = \Lambda := \mathrm{Circ}\left[M, -F, 0, \ldots, 0, -F^{\top}\right] \quad \overset{1:1}{\Leftrightarrow} \quad (M, F)$$

Covariance Selection Problem (A. P. DEMPSTER, 1972): The (ML) estimates of (M, F) are determined from $\hat{\Sigma}(0)$ and $\hat{\Sigma}(1)$ so that

$$\hat{\mathbf{\Sigma}} := \hat{\Lambda}^{-1} = \mathrm{Circ}\left[\hat{M}, -\hat{F}, 0, \ldots, -\hat{F}^{\top}\right]^{-1}$$

$\hat{\Lambda}_{i,j}$ must be zero exactly where the blocks $\Lambda_{i,j}$ are zero.

MATRIX COMPLETION PROBLEM: Having estimated the covariances $\hat{\Sigma}(0), \hat{\Sigma}(1)$ complete the block-Toeplitz matrix

$$\begin{bmatrix} \hat{\Sigma}(0) & \hat{\Sigma}(1) & \dots & ? & ? & ? \\ \hat{\Sigma}(1)^\top & \hat{\Sigma}(0) & \hat{\Sigma}(1) & \dots & ? & ? \\ \vdots & \hat{\Sigma}(1)^\top & \hat{\Sigma}(0) & \hat{\Sigma}(1) & \dots & ? \\ ? & \dots & \ddots & \ddots & \ddots & \\ ? & ? & \dots & \hat{\Sigma}(1)^\top & \hat{\Sigma}(0) & \hat{\Sigma}(1) \\ ? & ? & ? & \dots & \hat{\Sigma}(1)^\top & \hat{\Sigma}(0) \end{bmatrix}$$

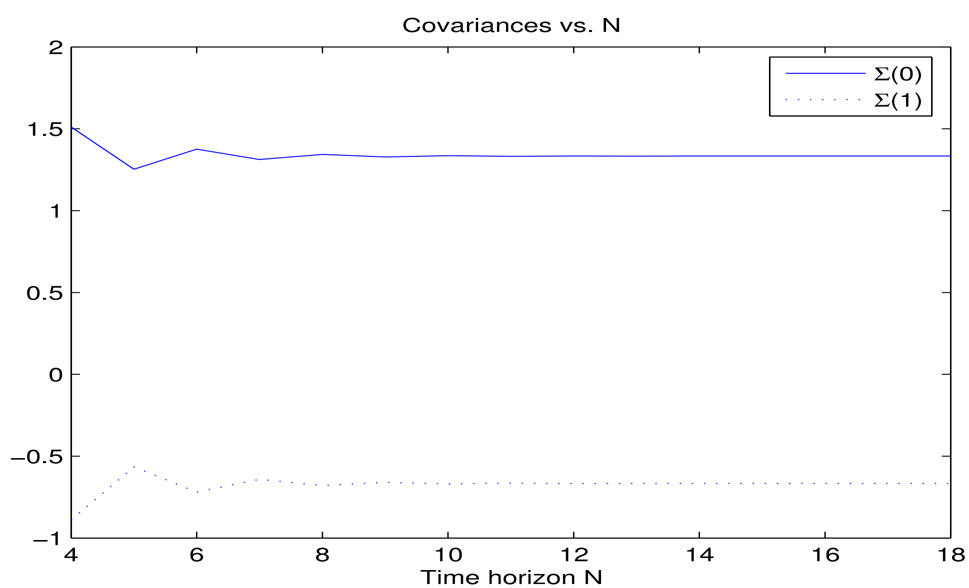
so that the inverse $\hat{\Lambda} = \hat{\mathbf{\Sigma}}^{-1}$ has *block-tridiagonal-circulant structure* (with constant entries).

Fact: [DEMPSTER] The completion problem has a unique solution.

ASYMPTOTICS OF THE COVARIANCE SELECTION PROBLEM

Dempster's original Algorithm is computationally intensive; it requires repeated inversion of matrices of size $O(N \times n)$. It does not seem to be useful for real size images. Need to resort to approximations.

For fixed M and F, the true covariances $\Sigma(0)$, $\Sigma(1)$ depend on N. As N increases the effect of the boundary condition vanishes: $\Sigma(0)$ and $\Sigma(1)$ converge to stationary values.


Covariances vs. N
Σ(0)
Σ(1)
2
1.5
1
0.5
0
−0.5
−1
4
6
8
10
12
14
16
18
Time horizon N

ASYMPTOTIC IDENTIFICATION

[B. Levy-1992]: *A stationary reciprocal process defined on the whole time axis* $\mathbb{Z}$ *(i.e.* $N = \infty$*) is a (stationary) Markov process. Can be described by a causal model*

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + \mathbf{v}(k)$$

The reciprocal model is *non causal* but both describe the same process. The parameters $\{M, F\}$ of a reciprocal model for a process *on the whole time axis* $\mathbb{Z}$ are in one-to-one correspondence with the parameters A and $Q = \mathrm{Var}\{\mathbf{v}\}$ of the Markov model. There are (invertible) functions Ψ_M, Ψ_F such that

$$M = \Psi_M(A, Q) \qquad F = \Psi_F(A, Q);$$

ASYMPTOTIC IDENTIFICATION (2)

From

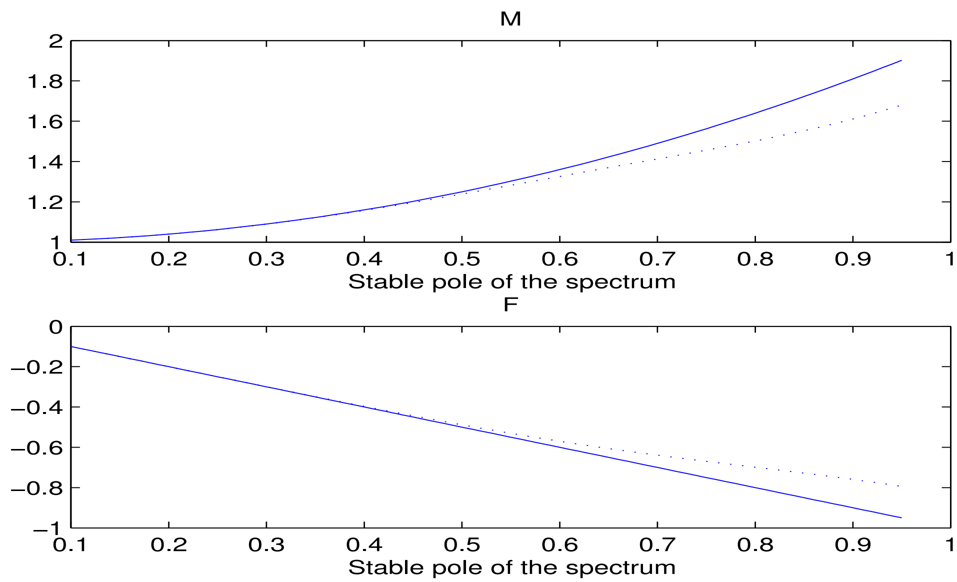
$$A = \Sigma(1)\Sigma(0)^{-1} \qquad Q = \Sigma(0) - A\Sigma(0)A^\top = \Sigma(0) - \Sigma(1)\Sigma(0)^{-1}\Sigma(1)^\top$$

get estimates $\hat{A}$, $\hat{Q}$; identify a stationary causal Markov model for $\mathbf{x}$. Get estimators $\hat{M}$, $\hat{F}$

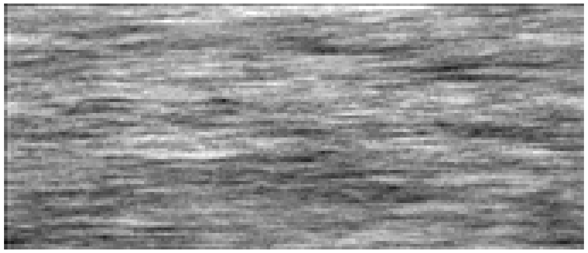
$$\hat{M} = \Psi_M(\hat{A}, \hat{Q}) \quad \hat{F} = \Psi_F(\hat{A}, \hat{Q});$$

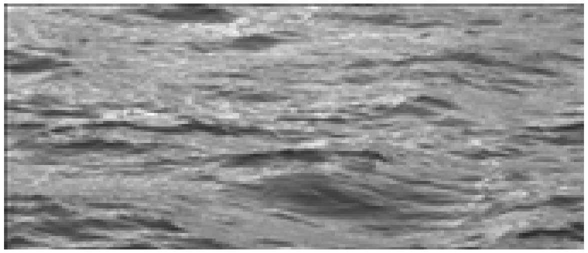
valid for $N = \infty$.

For N “large enough” are these close to maximum likelihood ?.



River sequence





INFINITE-INTERVAL COVARIANCE SELECTION PROBLEM

(Solved by the above trick) Is equivalent to a

matrix completion problem for an infinite block-Toeplitz >0 **matrix.**
Given $\Sigma(0), \Sigma(1)$, complete the infinite block-Toeplitz $\mathbf{\Sigma}$ so that

$$\mathbf{\Sigma}^{-1} = \Lambda := \begin{bmatrix} M, & -F & 0 & \dots & 0 & \dots \\ -F^\top & M & -F & 0 & \dots & \dots \\ 0 & -F^\top & M & -F & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \end{bmatrix} > 0$$

Well known! *Infinite band extension problem* [Dym,Gohberg, Kaashoek, ..]
In fact: **solvable by Levinson's algorithm!**

IDENTIFICATION OF NON-MINIMAL PROCESSES

Generalization: log-likelihood depends on $\nu + 1$ matrix parameters $\{M_k\}$

$$L(M_0, \ldots, M_\nu) = \log \det(\Lambda) - \sum_{k=0}^{\nu} \mathrm{Trace}\,\{M_k T_k(\underline{x})\}$$

ML estimated covariances

$$\begin{aligned}
\frac{1}{N} T_0 &= \hat{\Sigma}(0) = \text{M.L. Estimator of } \mathbb{E}\,\mathbf{x}(k)\mathbf{x}(k)^\top \\
&\cdots \\
\frac{1}{N} T_\nu &= \hat{\Sigma}(\nu) = \text{M.L. Estimator of } \mathbb{E}\,\mathbf{x}(k+\nu)\mathbf{x}(k)^\top
\end{aligned}$$

etc...

COVARIANCE SELECTION FOR NON-MINIMAL PROCESSES

Basic constraint:

$$\boldsymbol{\Sigma}^{-1} = \Lambda := \mathrm{Circ}\left[M_0, M_1, \ldots, M_\nu, 0, \ldots, 0, M_\nu^\top, M_{\nu-1}^\top, \ldots, M_1^\top\right] \quad \overset{1:1}{\Leftrightarrow} \quad (M_0, M_1, \ldots, M$$

Covariance Selection Problem (A. P. DEMPSTER, 1972): Compute the (unique) ML estimates of $(M_0, M_1, \ldots, M_\nu)$ from $\hat{\Sigma}(0) \ldots, \hat{\Sigma}(\nu)$, so that

$$\hat{\boldsymbol{\Sigma}}^{-1} = \mathrm{Circ}\left[\hat{M}_0, \hat{M}_1, \ldots, \hat{M}_\nu, 0, \ldots, 0, \hat{M}_\nu^\top, \hat{M}_{\nu-1}^\top, \ldots, \hat{M}_1^\top\right] = \hat{\Lambda}$$

MATRIX COMPLETION PROBLEM: Complete the estimated covariances $\hat{\Sigma}(0), \ldots, \hat{\Sigma}(\nu)$ with a sequence $\hat{\Sigma}(\nu+1), \ldots \hat{\Sigma}(N-1)$ in such a way that the inverse $\hat{\Lambda} = \hat{\boldsymbol{\Sigma}}^{-1}$ has *block-ν-banded-circulant structure*.

BAND EXTENSION BY LEVINSON

Given a finite positive definite sequence $\Sigma(0),\ldots,\Sigma(v)$ compute the Levinson polynomial of order v, $L_v(z)$. Approximate $AR(v)$ spectrum

$$\Phi_v(z) := L_v(z)^{-1}\Delta_v L_v(z^{-1})^{-\top}$$

has the same Fourier coefficients $\Sigma(0),\ldots,\Sigma(v)$ up to order v. The inverse

$$\Phi_v(z)^{-1} = L_v(z^{-1})^{\top}\Delta_v^{-1}L_v(z)$$

is a spectrum of the $MA(v)$ type. Since for *infinite* Toeplitz matrices

$$\mathbf{\Sigma}^{-1}(\Phi_v) = \mathbf{\Sigma}(\Phi_v^{-1})$$

$\mathbf{\Sigma}^{-1}$ is **banded** of width v!

CIRCULANT BAND EXTENSION BY LEVINSON

Spectral mapping

$$\mathbf{\Sigma}^{-1}(\Phi_\nu) = \mathbf{\Sigma}(\Phi_\nu^{-1})$$

does not hold for *finite* Toeplitz matrices. But it **holds for finite block-circulants**.

IDEA: do Levinson for DFT spectra!

CONCLUSIONS

- Stationary reciprocal state models : natural acausal models for processes on a finite lattice
- Apply naturally to textured images
- Elegant algebraic/frequency-domain treatment by block-circulant theory
- Exact (ML) model identification from data: open problem. Special case solvable by *circulant band extension*
- Much remains to be done !!!

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