



Information Flows and Distributed Control

Girish Nair
Dept. Electrical & Electronic Engineering
University of Melbourne

(based on work with Rob Evans)

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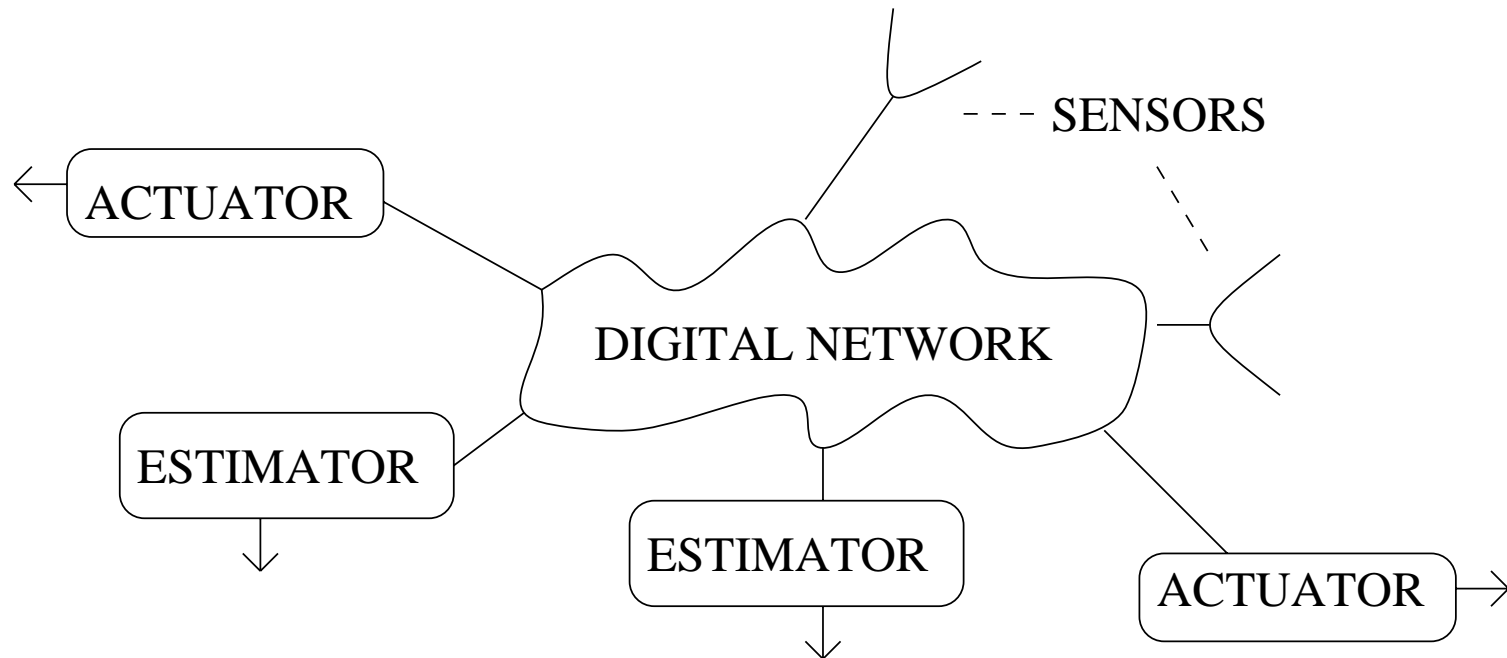


Outline

- Centralised Bit Rate-Limited Control
 - *Universal Performance Bounds*
- Cooperative Networked Control
 - *Irreducible Graph Cycles*
 - *Rate Regions for Stabilisability*

Networked Control Systems

E.g. *Micro-electromechanical* systems, mobile power control, irrigation, decentralised target tracking, manufacturing...

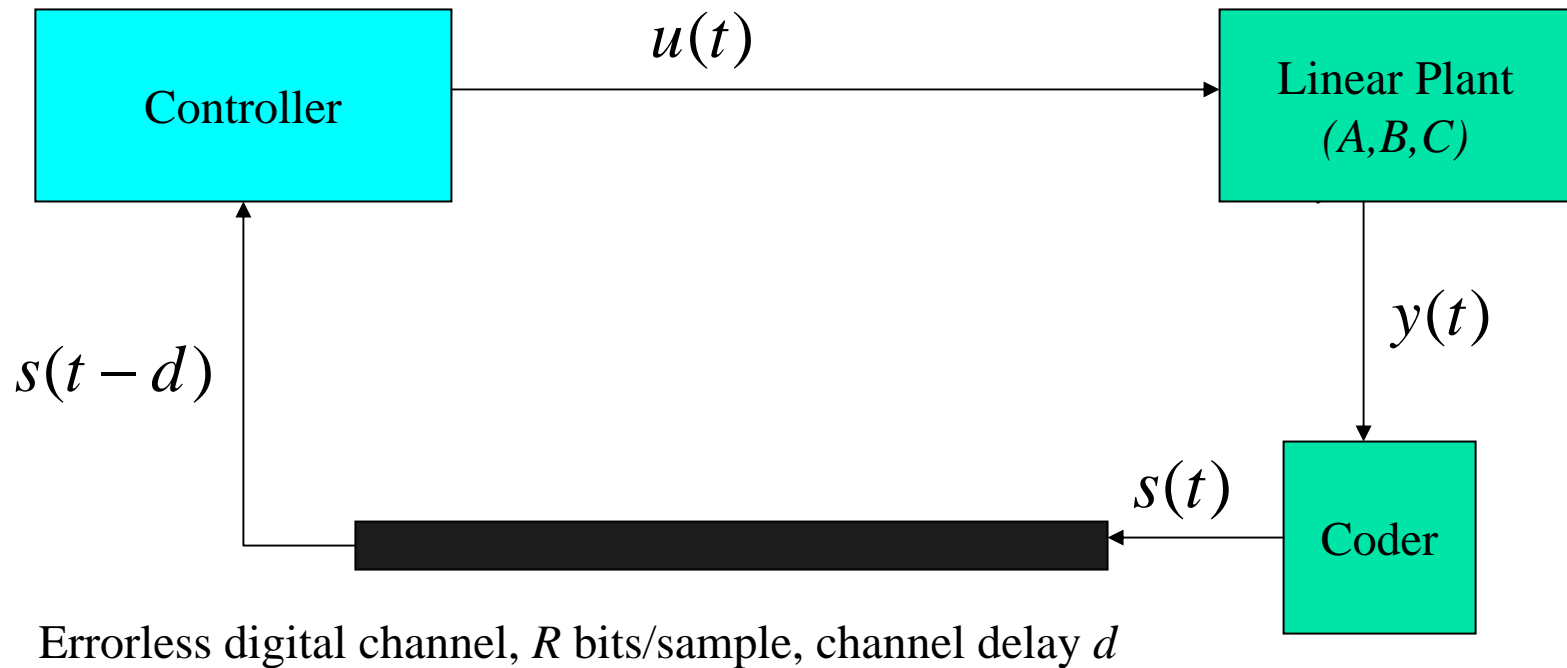




Communications Issues

- *Low data rates* → *Poor resolution*
- Long delays → Data less relevant
- Transmission errors → Data incorrect
- Shared Communication Medium Access

A Basic Question in Centralised Control



What is the smallest data rate for stabilisability?



Initial Work

- Early approach: analog-digital conversion errors in control = white noise.
- *Delchamps IEEE TAC '90*: A/D errors yield additional information.
LTI system stabilisable $\rightarrow 0$ with memoryless A/D, iff $|a| < 2$.
- *Wong & Brockett IEEE TAC '00*: scalar system stabilisable by memoryless quantisation & control iff

Feedback data rate $R > \log|a|$

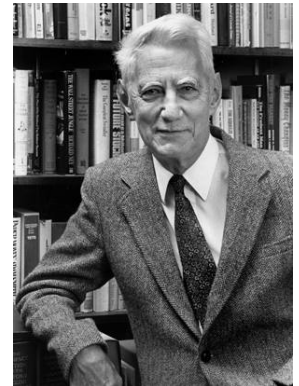
- Need for rigorous, coding-based approach at low data rates, rather than AWGN approximation

Source Coding Analogy

Shannon: An iid discrete source X can be 'reliably' communicated (arbitrarily small error probability) over a noiseless digital channel *iff*

$$R > h\{X\} := -E\{\log_2 p(X)\}.$$

- Guideline for allocating data rate.
- Quantifies source *information generation rate*.



In feedback control,
'reliability' = plant stability.





Performance Bounds for Centralised LTI Plants

Dynamics : $x(t+1) = Ax(t) + Bu(t) + v(t) \in \mathbf{R}^n$,

where $x(t)$ = state at time t , $u(t)$ = control input $\in \mathbf{R}^m$

$v(t)$ = additive dynamical noise,

dynamical & input matrices $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$.

Output : $y(t) = Cx(t) + w(t) \in \mathbf{R}^p$,

where $w(t)$ = additive output noise at time t ,

output matrix $C \in \mathbf{R}^{p \times n}$



Assumptions

$x(0) \in \mathbf{X}, v(t) \in \mathbf{V}, w(t) \in \mathbf{W}$ such that

A1 : The projection of \mathbf{V} onto unstable, f - dimensional eigenspace of A has f - dimensional volume > 0 .

I.e. the dynamical noise injects a finite amount of uncertainty into the unstable subspace

A2 : $\mathbf{X}, \mathbf{V}, \mathbf{W}$ bounded.



Coding and Control Laws

Continuous - valued plant outputs must be converted into symbols

$s_k \in$ finite alphabet \mathbf{S} for transmission.

Most generally, each symbol may be a causal, time - varying function of all past & present outputs and past symbols :

$$s(t) = \gamma(t, y(t), \dots, y(0), s(t-1), \dots, s(0)) \in \mathbf{S}.$$

Controller receives each symbol without error, after constant propagation delay d :

$$u(t) = \delta(t, s(t-d), \dots, s(0)) \in \mathbf{R}^m$$



Coder-Controller and Data Rate

$$\text{Coder - controller} := (\mathbf{S}, \gamma, \delta)$$

$$\text{Data Rate } R := \log_2 |\mathbf{S}| \quad (\text{bits/sample})$$

$$\text{Cost } J := \overline{\lim}_{t \rightarrow \infty} \sup_{x(0), \{v(k)\}, \{w(k)\}} \|x(t)\|$$

What can we say about the optimal control performance attainable over all coder - controllers with rate R ?

Universal Lower Bound on Performance vs. Rate & Delay

Any coder - controller which achieves $\lim_{t \rightarrow \infty} \sup_{x(0), \{v(k)\}, \{w(k)\}} \|x(t)\| < \infty$ must satisfy

$$\lim_{t \rightarrow \infty} \sup_{x(0), \{v(k)\}, \{w(k)\}} \|x(t)\| \geq \left(\frac{\text{vol}(\mathbf{V}^u)}{\beta} \right)^{1/f} \cdot \left(\frac{2^{Hd/f}}{1 - 2^{-(R-H)/f}} + \frac{2^{Hd/f} - 1}{2^{H/f} - 1} \right),$$

where $\mathbf{V}^u :=$ projection of disturbance set \mathbf{V} onto f - dimensional unstable subspace of A ,

$d \equiv$ channel delay, $H \equiv \sum_{\eta \in \sigma(A)} \log_2^+ |\eta|$, & $\beta :=$ unit sphere volume.

(Nair, Fagnani, Zampieri & Evans, Proceedings of the IEEE, 2007)

Properties of Performance Bound

- Worst case state increases with noise uncertainty.
& grows at least exponentially with delay d & H .

- **Low rate regime** $R \approx H$:

$$\lim_{t \rightarrow \infty} \sup_{x(0), \{v(k)\}, \{w(k)\}} \|x(t)\| \geq \left(\frac{\text{vol}(\mathbf{V}^u)}{\beta} \right)^{1/f} \frac{2^{Hd/f} f}{(R - H) \ln 2}.$$

- Worst case state $\rightarrow \infty$ as $R - H \downarrow 0$.

- **Classical regime** $R \gg H$:

$$\lim_{t \rightarrow \infty} \sup_{x(0), \{v(k)\}, \{w(k)\}} \|x(t)\| \geq \left(\frac{\text{vol}(\mathbf{V}^u)}{\beta} \right)^{1/f} \left(\frac{2^{H(d+1)/f} - 1}{2^{H/f} - 1} \right).$$

- Applies to *any* causal, possibly nonlinear & time - varying control law.



Tightness of Bound

For a fully - observed scalar plant

$$x(t + 1) = ax(t) + bu(t) + v(t) \in \mathbf{R}, \quad y(t) = x(t),$$

bound is attainable! I.e.

$$J_{\text{optimal}} \equiv \min_{\gamma, \delta} \left\{ \overline{\lim}_{t \rightarrow \infty} \sup_{x(0), \{v(k)\}, \{w(k)\}} |x(t)| \right\} = \frac{\text{vol}(\mathbf{V})}{2} \left(\frac{2^{Hd}}{1 - 2^{-(R-H)}} + \frac{2^{Hd} - 1}{2^H - 1} \right).$$

Corollary : Infimum cost over all nonlinear & time - varying control laws without rate constraints (i.e. $R \rightarrow \infty$) is

$$J_* = \frac{\text{vol}(\mathbf{V})}{2} \frac{2^{H(d+1)} - 1}{2^H - 1}$$



Key Properties used in Proof

1. \forall Lebesgue - measurable $\mathbf{K} \subset \mathbf{R}^f$,

$$\text{vol}(\mathbf{K}) \leq \beta \sup_{x \in \mathbf{K}} \|x\|^f \quad (= \text{vol. of bounding sphere}).$$

\Rightarrow Worst case state norm is lower - bounded by volume.

2. $\text{vol}(A\mathbf{K}) = |\det A| \text{vol}(\mathbf{K}), \text{vol}(\mathbf{K} + u) = \text{vol}(\mathbf{K})$

\Rightarrow Plant dynamics multiply uncertainty volume, additive control does not change it.

3. Brunn - Minkowski Inequality :

$$\text{vol}(\mathbf{J} + \mathbf{K})^{1/f} \geq \text{vol}(\mathbf{J})^{1/f} + \text{vol}(\mathbf{K})^{1/f}, \quad \text{where } \mathbf{J} + \mathbf{K} := \{x + y : x \in \mathbf{J}, y \in \mathbf{K}\}.$$

\Rightarrow Additive disturbances have super - additive effect on f - root uncertainty volume.



Optimal Quadratic Regulation (Scalar Case)

Noiseless, fully - observed scalar plant

$$X(t+1) = aX(t) + bU(t) \in \mathbf{R}^n, \quad \forall t \in \mathbf{Z}_+,$$

where $E\{|X(0)|^{2+\varepsilon}\} < \infty$, for some $\varepsilon > 0$.

Assume channel delay $d = 1$ & consider quadratic regulation cost

$$J_{\text{QR}} := \sum_{t=0}^{\infty} E\{X(t+1)^2 + rU(t)^2\}, \quad r > 0.$$



Optimal Quadratic Regulation with Data Rate Constraints

If X_0 is uniformly distributed, then for any data rate $R > \log_2 |a|$,

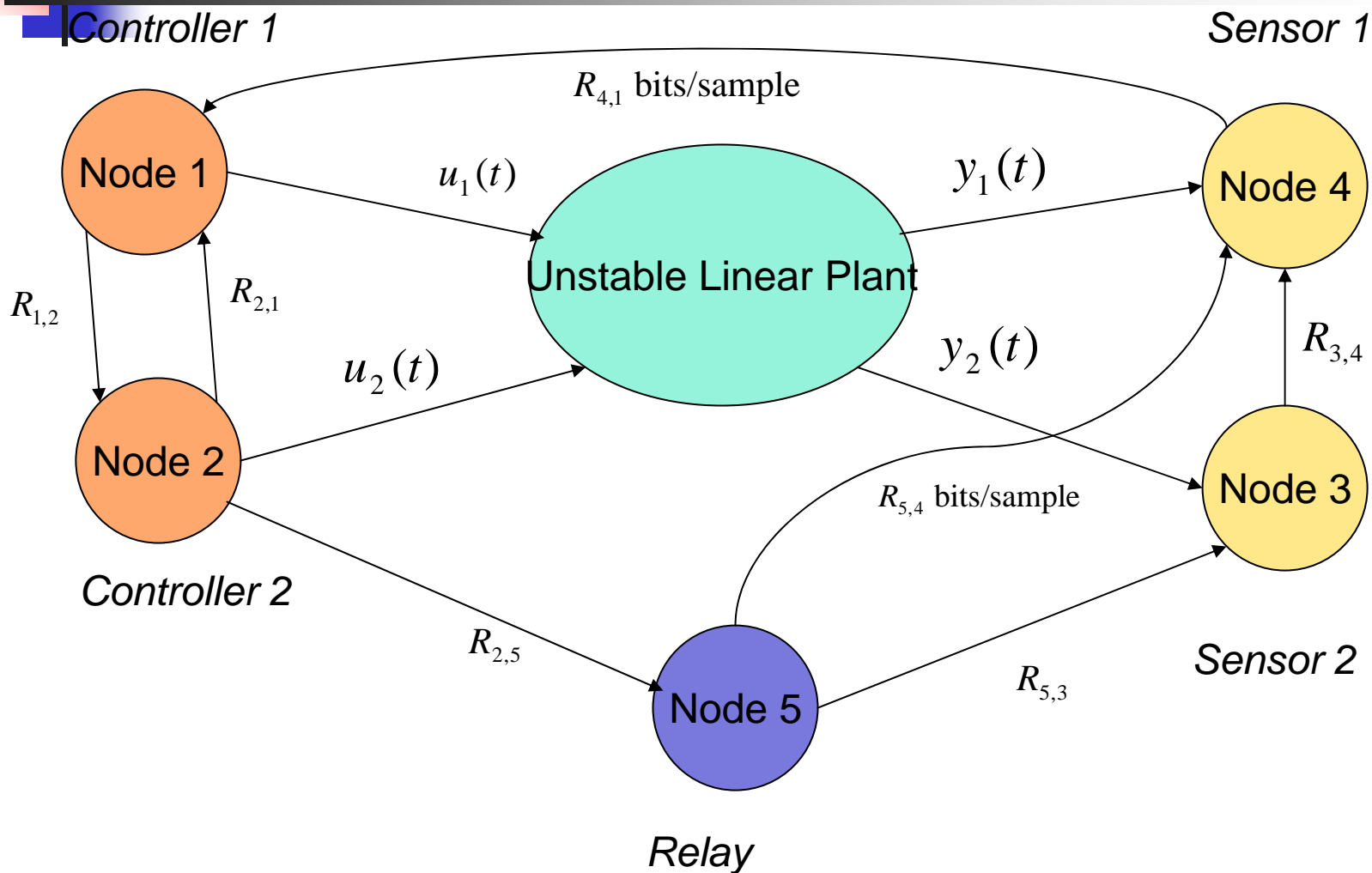
$$\min_{\gamma, \delta} J_{QR} = J_{\text{classical}} + \frac{a^2(p+1) - p}{(2^{2R}/a^2) - 1} \text{var}\{X_0\},$$

where p solves the Riccati equation $p = \frac{a^2(p+1)r}{b^2(p+1) + r}$,

$J_{\text{classical}} := \text{Expected classical cost} = a^2(p+1) \text{var}\{X_0\} + p\text{E}\{X_0\}^2$.

(Proc. 14th IFAC Symp. System Identification, 2006)

Cooperative Networked Control





Formulation

Plant has U inputs, Y outputs:

$$x(t+1) = Jx(t) + \sum_{i=1}^U B_i u_i(t) \in \mathbf{R}^n,$$

$$y_j(t) = C_j x(t) + w(t) \in \mathbf{R}^{q_j}, \quad j = 1, \dots, Y.$$

A1: $x(0)$ is unknown & can take any value in a bounded set \mathbf{X}

with positive volume in \mathbf{R}^n

A2: $w(t)$ can take any value in a bounded set $\mathbf{W} \subset \mathbf{R}^{q_j}$

A3: The plant is controllable from all inputs together & observable from all outputs together.

A4: J has real & distinct eigenvalues.



Coding and Control Scheme

Let every ordered pair of distinct nodes (q, r) be connected by a errorless, unidirectional digital channel $(q \rightarrow r)$ that at each time instant t transports one symbol $s_{q,r}(t)$ from a finite, time - varying alphabet $\mathbf{S}_{q,r}(t)$.

$$\text{(Average) Channel Data Rate } R_{q,r} := \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^{t-1} \log_2 |\mathbf{S}_{q,r}(k)| \text{ (bits/sample)}$$

No direct communication from $(q \rightarrow r) \Rightarrow \text{set } |\mathbf{S}_{q,r}(k)| = 1 \text{ so that } R_{q,r} = 0.$

Perfect communication from $(q \rightarrow r) \Rightarrow \text{set } |\mathbf{S}_{q,r}(k)| \rightarrow \infty \text{ so that } R_{q,r} \rightarrow \infty.$



Coding Laws

Let $S(t) := [s_{q,r}(t)]_{1 \leq q,r \leq N}$ be the $N \times N$ matrix of all channel symbols transmitted at time t , with 0's on the diagonal.

The vector of all channel symbols received at time t by the r th node = r th column $s_r(t)$.

Each q th node uses an encoding function $\gamma_{q,r}(t, \cdot)$ to transmit onto channel ($q \rightarrow r$).

If q th node = j th sensor, $s_{q,r}(t) \equiv \gamma_{q,r}\left(t, \{y_j(k)\}_{k=0}^t, \{s_q(k)\}_{k=0}^t\right) \in \mathbf{S}_{q,r}(t)$

If not, $s_{q,r}(t) \equiv \gamma_{q,r}\left(t, \{s_q(k)\}_{k=0}^t\right) \in \mathbf{S}_{q,r}(t)$



Coding and Control Scheme

If q th node is i th controller, it applies $u_i(t) \equiv \delta_i\left(t, \{s_q(k)\}_{k=0}^t\right) \in \mathbf{R}^{m_i}$.

If q th node is simultaneously i th controller & j th sensor, $u_i(t) \equiv \delta_i\left(t, \{s_q(k)\}_{k=0}^t, \{y_j(k)\}_{k=0}^t\right)$.

Cooperative Networked Controller (CoNC) $\mathbf{C} := \left\{ \left(\mathbf{S}_{q,r}(\cdot), \gamma_{q,r}(\cdot), \delta_i(\cdot) \right) \right\}_{q,r,i}$.

Let $R := \left[R_{q,r} \right]_{1 \leq q,r \leq N} \in \mathbf{R}^{N \times N}$, with 0's on the diagonal.

Question : how can one characterise the achievable region of rate matrices R at which there exists a CoNC \mathbf{C} that uniformly stabilises the plant state,

$$\sup_{t, x(0) \in \mathbf{X}, w_j(k) \in \mathbf{W}_j} \|x(t)\| < \infty?$$



Previous Literature

- Classical LTI analyses without bit-rate constraints (Corfmat & Morse, Automatica '76, Vaz & Davison, IEEE TAC '73)
- LTV solution (Anderson & Moore, IEEE TAC '81)

Bit-rate-constrained formulations:

- Noiseless plant, multiple sensors & one controller
 - Sufficient condition (Tatikonda 2000)
 - Necessary & sufficient condition (Matveev & Savkin, SIAM J. Con. Opt. 2006)
- Noiseless plant, multiple sensors & controllers
 - Separate necessary & sufficient conditions(Nair, Evans & Caines, CDC'04)
 - Necessary & sufficient condition (Matveev & Savkin, CDC'05)

Controllability & Observability Structure

Rate of information generation is determined by plant dynamical modes.
→ Need to understand how modes affect outputs & are affected by inputs.

Let $x_h := h$ th component, or mode, of x .

For i th controller, let $d_{i,h} := \begin{cases} 1 & \text{if } h\text{th row of input matrix } B_i \text{ is } \neq 0 \\ 0 & \text{otherwise} \end{cases}$.

For j th sensor, let $e_{h,j} := \begin{cases} 1 & \text{if } h\text{th column of output matrix } C_j \text{ is nonzero} \\ 0 & \text{otherwise} \end{cases}$.



Graph Cycles

Let an *irreducible cycle* \mathbf{c} for mode x_h be any finite sequence of nodes and modes s.t.

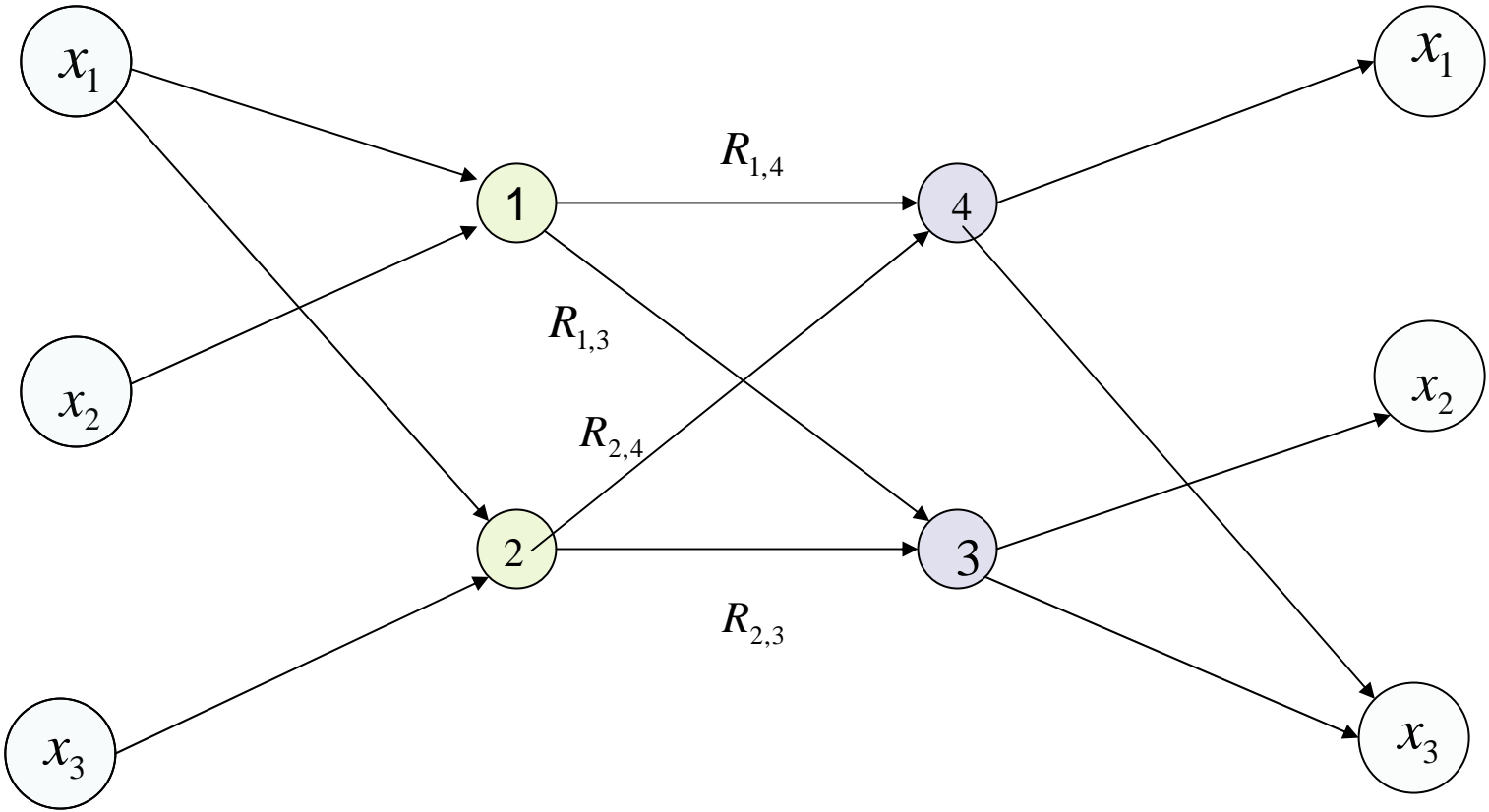
- 1) x_h begins and ends the sequence and does not occur elsewhere in it.
- 2) Every other mode or node can only occur once.
- 3) The successor of any mode x_k in the sequence is a sensor node which is affected by it, $e_{k,j} = 1$
- 4) The predecessor of any mode x_k in the sequence is a controller node which can affect it, $d_{i,k} = 1$.

Define \mathbf{C}_h to be the set of all irreducible x_h - cycles. Clearly, $|\mathbf{C}_h| < \infty$.

Call a cycle *trivial* if it passes over any channel with zero rate.

Observability Structure

Controllability Structure



Example of non-trivial irreducible X3-cycles:
 ($X_3, 2, 4, X_3$), ($X_3, 2, 3, X_3$), ($X_3, 2, 4, X_1, 1, 3, X_3$), ($X_3, 2, 3, X_2, 1, 4, X_3$)

Main Result: Characterisation of Achievable Rate Matrices

If a cooperative networked controller (CoNC) achieves uniform stability with channel rate matrix $R = [R_{q,r}]$, then for each unstable open-loop eigenvalue η_h & irreducible x_h -cycle \mathbf{c} , $\exists \rho_{h,\mathbf{c}} \geq 0$ s.t.

$$R_{q,r} \geq \sum_{h,\mathbf{c} \in \mathbf{C}_h: (q \rightarrow r) \in \mathbf{c}} \rho_{h,\mathbf{c}}, \quad \forall \text{ channels } (q \rightarrow r).$$
$$\sum_{\mathbf{c} \in \mathbf{C}_h} \rho_{h,\mathbf{c}} \geq \log_2 |\eta_h|.$$

Conversely, if the strict forms of these inequalities are feasible for some rate matrix R' , it is possible to explicitly construct a CoNC that achieves uniform stability at channel rates $R_{q,r} \leq R'_{q,r} + \varepsilon$, for arbitrary ε . (Nair & Evans, submitted 2007)

Interpretation as Subchannel Rates

Each auxiliary variable $\rho_{h,c} \equiv$ that portion of the available channel rates along the irreducible x_h - cycle \mathbf{c} that may be equivalently associated with $x_h(0)$.

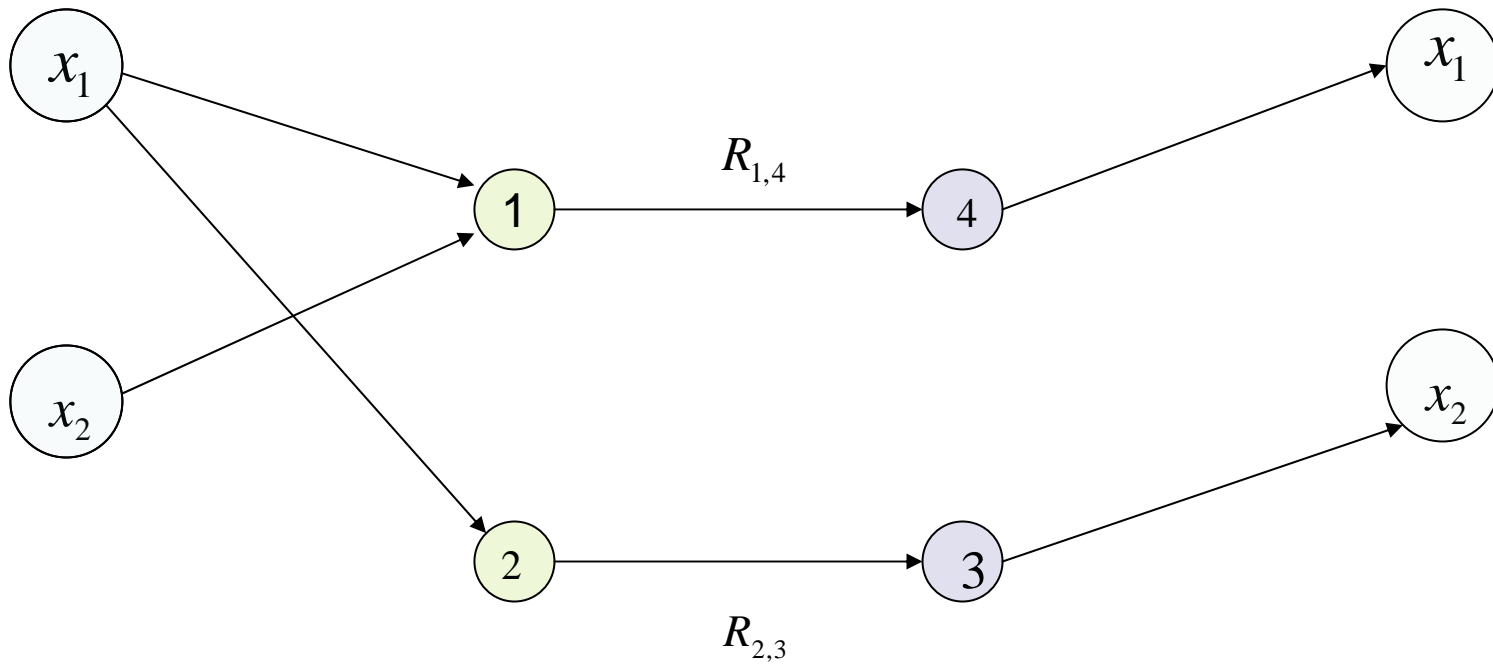
More precisely, for a given stabilising CoNC, $\rho_{h,c} = \min_{(q \rightarrow r) \in \mathbf{c}} \liminf_{t \rightarrow \infty} \log_2 \left(\frac{\text{vol}\{x_h(0)\}}{\max_{\{s_{q,r}(k)\}_{k=0}^t} \text{vol}\{x_h(0) \mid \{s_{q,r}(k)\}_{k=0}^t\}} \right)$.

i.e. the smallest or bottle - neck rate at which the maximum uncertainty volume in the h th initial mode is reduced by each coder in the cycle.

Irreducible x_h - cycles may be thought of as independent pathways via which information about $x_h(0)$ is carried from sensors observing it to controllers which can affect it.

These pathways may contain one or more other modes \equiv Signalling through the plant.

x_h - cycles that are not irreducible can be ignored, since the repetition of a node or mode does not increase the flow of information.



Example. Non-trivial irreducible X_1 -cycles: $(X_1, 1, 4, X_1)$, $(X_1, 2, 3, X_2, 1, 4, X_1)$
 Non-trivial irreducible X_2 -cycle: $(X_2, 1, 4, X_1, 2, 3, X_2)$



Example: Stabilisability Criterion

For stabilisability to be possible, there must exist $\rho_{1,c1}, \rho_{1,c2}, \rho_{2,c3} \geq 0$ s.t.

$$\rho_{1,c1} + \rho_{1,c2} + \rho_{2,c3} \leq R_{1,4}$$

$$\rho_{1,c2} + \rho_{2,c3} \leq R_{2,3}$$

$$\rho_{1,c1} + \rho_{1,c2} \geq \log_2 |\eta_1|$$

$$\rho_{2,c3} \geq \log_2 |\eta_2|$$



Comparison with Decentralised LTV Control

In classical decentralised control, the channel rates between sensors & controllers are either 0 (no connection) or ∞ (perfect communication).

Recall that the plant is controllable over all available inputs & observable over all available outputs. Then for every unstable mode x_h , there must be at least one sensor that can observe it & one controller that can affect it.

\Rightarrow As the number of nodes & modes is finite, for every unstable mode x_h there must then exist at least one non-trivial, irreducible x_h -cycle \mathbf{c} , i.e. which passes over channels with nonzero rate.

\Rightarrow Since any non-zero rate $= \infty$, set $\rho_{h,c} = \infty$ for each such cycle, satisfying all stabilisability conditions.

Thus, decentralised stability is possible by some possibly non-linear or time-varying law, agreeing with LTV controller results from Anderson & Moore '81.



Necessity Argument

Two key techniques

1. $2 \sup \{ \|x(t)\| \} \geq \max_{\{S(k)\}_{k=0}^t} \text{vol}\{x_h(t) | \{S(k)\}_{k=0}^t\} \geq |\eta_h|^t \max_{\{S(k)\}_{k=0}^t} \text{vol}\{x_h(0) | \{S(k)\}_{k=0}^t\} \equiv |\eta_h|^t v_h(t).$

⇒ Instead of state magnitudes, analyse evolution of maximum conditional uncertainty *volume*, for each initial mode.

2. In the absence of noise, the encoders in the given CoNC may be replaced by separate quantisers for each initial mode, operating at rates $\rho_{h,c}$, without increasing $v_h(t)$.

Application of arguments for scalar centralised stabilisation to each mode then yields necessity.



Stabilising Scheme

1) Divide time into coding cycles of length T . At each time $t = lT$, let χ_l be a globally known bound on $\|x(lT)\|$.

2) 1st n_1 instants in each cycle = plant signalling phase. Controllers communicate data bits through the plant by sending outputs to a finite number of pre - specified levels.

These levels are sufficiently far apart to be perfectly recoverable despite bounded measurement noise & uncertain plant state.

3) Next n_2 instants = mode estimation & coding phase. Plant is allowed to evolve in open - loop.

Due to the unstable dynamics, arbitrarily accurate estimates $\hat{x}_h(lT)$ may be obtained.

Binary expansion of $\frac{\hat{x}_h(lT)}{2\chi_l} + 0.5 \in [0,1)$ is computed & different blocks of bits are transmitted onto the irreducible cycles.

4) Stabilisation phase. Controllers reassemble the bit blocks received over each irreducible cycle and apply inputs to stabilise the modes.



Further Work

- Dynamical noise
- Performance Bounds
- Delays
- Transmission Errors