An (In)complete Kalman Decomposition for Uncertain Linear Systems

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Introduction

- This paper considers the structure of uncertain linear systems building on concepts of “robust unobservability and “possible controllibility”.
- One reason for considering the issue of observability for uncertain systems might be to determine if a robust state estimator can be constructed for the system. Similarly, one might consider the issue of controllability to determine if a robust state feedback controller for the system.
- In this case, one would be interested in the question of whether the system is “observable” or “controllable” for all possible values of the uncertainty.
- The notions of controllability and observability are also central to realization theory. For example, it is known that if a linear time invariant system contains unobservable states or uncontrollable, those states can be removed in order to obtain a reduced dimension realization of the system’s input-output behavior.

- For the case of uncertain systems when one is interested in the issue of “minimal realization”, a natural extension of this notion of observability is to consider robustly unobservable states which are “unobservable” for all possible values of the uncertainty.
- Similarly, a natural extension of the notion of controllability is to consider “possibily controllable” states which are “controllable” for some possible value of the uncertainty.
- We formally define these notions of robust unobservability and possible controllability in terms of certain constrained optimization problems.
- We then apply the S-procedure to obtain conditions in terms of unconstrained LQ optimal control problems dependent on Lagrange multiplier parameters.

- We then develop a geometric characterization for the set of robustly unobservable states. We also (partially subject to some conditions) develop a geometric characterization of the set of possibly controllable states
- These characterizations imply that the set of robustly unobservable states is in fact a linear subspace.
- Similarly (under some conditions), we show that the set of possibly controllable states is a linear subspace.
- These characterizations leads to a Kalman type decomposition for the uncertain systems under consideration (provided the required conditions are satisfied).

Problem Formulation

- We consider the following linear time invariant uncertain system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 u(t) + B_2 \xi(t); \\ z(t) &= C_1 x(t) + D_1 u(t); \\ y(t) &= C_2 x(t) + D_2 \xi(t) \end{aligned}$$

- $x \in \mathbf{R}^n$ is the *state*, $u \in \mathbf{R}^m$ is the *control input*, $y \in \mathbf{R}^l$ is the *measured output*, $z \in \mathbf{R}^h$ is the *uncertainty output*, and $\xi \in \mathbf{R}^r$ is the *uncertainty input*.

- The system uncertainty is described by an integral quadratic constraint on the uncertainty input.

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Integral Quadratic Constraint

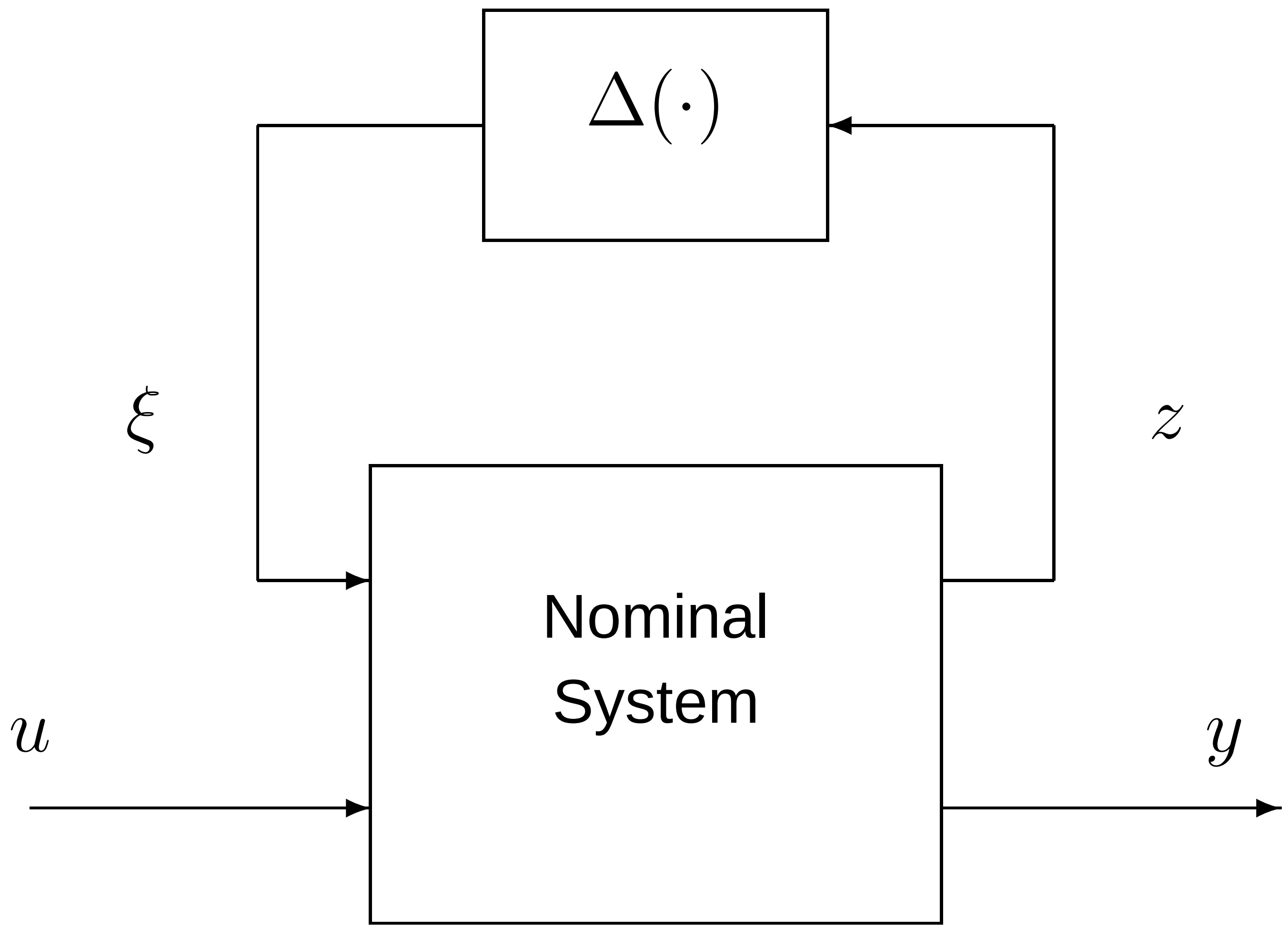
- On a time interval $[0, T]$, we consider uncertainty inputs $\xi(\cdot) \in \mathbf{L}_2[0, T]$ such that for any control input $u(\cdot) \in \mathbf{L}_2[0, T]$ and any corresponding solution $x(\cdot)$ to the system state equations defined on $[0, T]$, then $\xi(\cdot) \in \mathbf{L}_2[0, T]$, and

$$\int_0^T \left(\|\xi(t)\|^2 - \|z(t)\|^2 \right) dt \leq d$$

where $d > 0$ is a given constant.

- The class admissible uncertainty inputs is denoted $\mathbf{\Xi}$.

The uncertain system can be represented by the following block diagram.



- Our fundamental question:
  - Given such an uncertain system model, can we construct a simpler uncertain system model with smaller state dimensions in the state equations such that it will realize the same set of input-output behaviours as the original uncertain system model.
  - The corresponding problem in linear systems theory is given a state space model, construct a state space model of smaller state dimension (minimal realization) with the same transfer function matrix.
  - Based on an analogy with the linear time invariant systems result, our results provide a candidate method of achieving such reduced order models for uncertain linear systems. Verification that our results in fact lead to these reduced dimension realizations for uncertain systems could be verified using some other results and is the subject of future research.

Definition. *The* robust unobservability function *for the above uncertain system on* $[0,T]$ *is defined as*

$$L_o(x_0,T) \triangleq \sup_{\xi(\cdot)\in\Xi} \int_0^T \|y(t)\|^2 dt$$

where $x(0) = x_0$.

- This definition extends the standard definition of the observability Gramian for linear time invariant systems.

Notation.

$$\mathcal{D} \triangleq \{d \in \mathbb{R} : d > 0\}.$$

Definition. *A non-zero state* $x_0 \in \mathbf{R}^n$ *is said to be* robustly unobservable *on* $[0, T]$ *if*

$$\inf_{d \in \mathcal{D}} L_o(x_0, T) = 0.$$

A non-zero state $x_0 \in \mathbf{R}^n$ *is said to be* (differentially) robustly unobservable *if it is robustly unobservable on* $[0, T]$ *for all sufficiently small* $T > 0$.

The set of all differentially robustly unobservable states is referred to as the robustly unobservable set $\mathcal{U}$.

- The set $\mathcal{U}$ is analogous to the unobservable subspace in linear time invariant systems theory.

Definition. *The* possible controllability function *for the uncertain system defined on the time interval* $[0,\ T]$ *is defined as*

$$L_c(x_0, T) \triangleq$$

$$\sup_{\epsilon > 0} \inf_{\xi(\cdot) \in \Xi} \inf_{u(\cdot) \in \mathbf{L}_2[0,T]} \left[\frac{\|x(-T)\|^2}{\epsilon} + \int_0^T \|u(t)\|^2 dt \right]$$

where $x(0) = x_0$.

- This definition extends the standard definition of the controllability Gramian for linear time invariant systems.
- This definition is more complicated that the corresponding observability definition due to the requirement to enforce a terminal condition on the state.

Definition. *A non-zero state* $x_0 \in \mathbf{R}^n$ *is said to be* possibly controllable on $[0,\ T]$ *if*

$$\sup_{d \in \mathcal{D}} L_c(x_0, T) < \infty.$$

A non-zero state $x_0 \in \mathbf{R}^n$ *is said to be* (differentially) possibly controllable *if it is possibly controllable on* $[0,\ T]$ *for all sufficiently small* $T > 0$*.*

The set of all differentially possibly controllable states is referred to as the possibly controllable set $\mathcal{C}$*.*

- The set $\mathcal{C}$ is analogous to the controllable subspace in linear time invariant systems theory.

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Optimal Control Characterizations

- We can apply the S-procedure to provide characterizations of the *possibly controllable set* and the *robustly unobservable set* in terms of certain LQ optimal control problems and corresponding Riccati differential equations.

An Unconstrained Optimization Problem for Robust Unobservability

- We define a function $V_\tau(x_0, T)$ as follows:

$$V_\tau(x_0,T) \triangleq \inf_{\xi(\cdot)\in \mathbf{L}_2[0,T]} \int_0^T \begin{pmatrix} -\|y\|^2 \\ +\tau\|\xi\|^2 - \tau\|z\|^2 \end{pmatrix} dt.$$

- Here $\tau \geq 0$ is a given constant.

Observation. *Note that by setting $\xi(\cdot) \equiv 0$, we can see that*

$$V_\tau(x_0,T) \leq 0 \; \forall \tau \geq 0.$$

Since $V_\tau(x_0,T)$ is the infimum of a collection of functions which are affine linear in τ, then $V_\tau(x_0,T)$ must be a concave function of τ.

Theorem. *A state $x_0 \in \mathbf{R}^n$ robustly unobservable on $[0, T]$ if and only if*

$$\sup_{\tau \geq 0} \{V_\tau(x_0, T)\} = 0$$

- We can calculate $V_\tau(x_0, T)$ by using a Riccati equation approach to solving the corresponding optimal control problem.

Theorem. *Let $\tau > 0$ be given such that*

$$\tau I - D_2' D_2 > 0.$$

Then

$$V_\tau(x_0, T) > -\infty \; \forall x_0 \in \mathbf{R}^n$$

if and only if the Riccati differential equation

$$\begin{aligned} -\dot{Q} &= A'Q + QA \\ &\quad -(QB_2 - C_2'D_2)\left[\tau I - D_2'D_2\right]^{-1}(B_2'Q - D_2'C_2) \\ &\quad -C_2'C_2 - \tau C_1'C_1; Q(T) = 0 \end{aligned}$$

has a solution $Q_\tau(t)$ defined on $[0, T]$. In this case,

$$V_\tau(x_0, T) = x_0' Q_\tau(0) x_0.$$

A family of Unconstrained Optimal Control Problems for Possible Controllability

- For the uncertain system defined on the time interval $[0,\ T]$, we define functions $W_\tau^\epsilon(x_0, T)$ and $W_\tau(x_0, T)$ as follows for $\tau \geq 0$:

$$W_\tau^\epsilon(x_0, T) \triangleq \inf_{[\xi(\cdot),u(\cdot)]\in\mathbf{L}_2[0,T]} \frac{\|x(T)\|^2}{\epsilon} + \int_0^T \left(\|u\|^2 + \tau\|\xi\|^2 - \tau\|z\|^2 \right) dt$$

subject to $x(0) = x_0$;

$$W_\tau(x_0, T) \triangleq \sup_{\epsilon>0} W_\tau^\epsilon(x_0, T).$$

- Again the controllability case is more complicated than the observability case due to the terminal constraint on the state.

Theorem. *A non-zero state $x_0 \in \mathbf{R}^n$ is possibly controllable on $[0, T]$ if and only if*

$$\sup_{\epsilon > 0} \sup_{\tau \geq 0} W_\tau^\epsilon(x_0, T) = \sup_{\tau \geq 0} W_\tau(x_0, T) < \infty$$

- We can calculate $W_\tau^\epsilon(x_0, T)$ by using a Riccati equation approach to solving the corresponding optimal control problem.

Theorem. *Let $\tau > 0$ be such that $I - \tau D_1' D_1 > 0$. Then*

$$W_\tau^\epsilon(x_0, T) > -\infty \; \forall x_0 \in \mathbf{R}^n$$

if and only if the Riccati differential equation

$$\begin{aligned} -\dot{P}^\epsilon = \\ & A' P^\epsilon + P^\epsilon A \\ & -(P^\epsilon B_1 - \tau C_1' D_1)(I - \tau D_1' D_1)^{-1}(P^\epsilon B_1 - \tau C_1' D_1)' \\ & -\frac{P^\epsilon B_2 B_2' P^\epsilon}{\tau} - \tau C_1 C_1'; \qquad P^\epsilon(T) = I/\epsilon \end{aligned}$$

has a solution $P_\tau^\epsilon(t)$ defined on $[0, T]$. In this case,

$$W_\tau^\epsilon(x_0, T) = x_0' P_\tau^\epsilon(0) x_0.$$

In order to calculate $W_\tau(x_0, T)$ we will also consider the following Riccati Differential Equations:

$$\dot{S}^\epsilon = AS^\epsilon + S^\epsilon A' - (B_1 - \tau S^\epsilon C_1' D_1)(I - \tau D_1' D_1)^{-1}(B_1 - \tau S^\epsilon C_1' D_1)' - \frac{B_2 B_2'}{\tau} - \tau S^\epsilon C_1 C_1' S^\epsilon; \qquad S^\epsilon(T) = \epsilon I;$$

$$\dot{S} = AS + SA' - (B_1 - \tau S C_1' D_1)(I - \tau D_1' D_1)^{-1}(B_1 - \tau S C_1' D_1)' - \frac{B_2 B_2'}{\tau} - \tau S C_1 C_1' S; \qquad S(T) = 0$$

which are solved backwards in time.

Theorem. *Let* $\tau > 0$ *be such that* $I - \tau D_1' D_1 > 0$*. Also suppose there exists an* $\epsilon_0 > 0$ *such that for all* $\epsilon \in (0, \epsilon_0)$*, all non-zero* $x_0 \in \mathbf{R}^n$ *then* $W_\tau^\epsilon(x_0, T) > 0$*. Then for any* $\epsilon \in (0, \epsilon_0)$*, the above Riccati equations have solutions* $S_\tau^\epsilon(t) > 0$ *and* $S_\tau(t) \geq 0$ *defined on* $[0, T]$ *and for any* $x_0 \neq 0$

$$W_\tau^\epsilon(x_0, T) = x_0' \left[S_\tau^\epsilon(0)\right]^{-1} x_0 > 0.$$

Also, if $S_\tau(0) > 0$ *then*

$$W_\tau(x_0, T) = x_0' \left[S_\tau(0)\right]^{-1} x_0 > 0.$$

Furthermore, if the matrix $S_\tau(0) \geq 0$ *is singular and* x_0 *is not contained within the range space of* $S_\tau(0)$*, then*

$$W_\tau(x_0, T) = \infty.$$

Notes

- Although the result is useful in proving our results, it suffers from the difficulty that the Riccati equation for S_τ may have a finite escape even if $W_\tau(x_0, T)$ remains finite.
- For all of the Riccati equations being considered, we can choose the time interval $[0, T]$ sufficiently small to ensure that there exists a solution to the Riccati equation on that interval at least for a given value of τ.
- The Riccati equation for S^ϵ is obtained from the Riccati equation for P^ϵ by making the substitution $S^\epsilon = [P^\epsilon]^{-1}$.
- The Riccati equation for S is obtained from the Riccati equation for S^ϵ by taking the limit as $\epsilon \longrightarrow 0$.

- The Riccati equation for S corresponds to the Riccati equation for Q for the dual system

$$
\begin{aligned}
\dot{x} &= -A'x + C_1'\xi; \\
y &= B_1'x - D_1'\xi; \\
z &= B_2'x
\end{aligned}
$$

- This suggests a duality between robust observability and possible controllability. However, technical difficulties arise if the Riccati equation for S_τ does not have a positive definite solution or has a finite escape time.

Geometric Results on Robust Unobservability

Lemma. *If the time interval $[0,T]$ is chosen sufficiently short then following statements are equivalent:*

1. *There exists an $x_0 \in \mathbf{R}^n$ such that the supremum in $\sup_{\tau \geq 0} V_\tau(x_0)$ is achieved at $\tau = 0$.*
2. *The transfer function from input ξ to output y is zero; i.e.,*

$$G(s) \triangleq C_2(sI - A)^{-1}B_2 + D_2 \equiv 0.$$

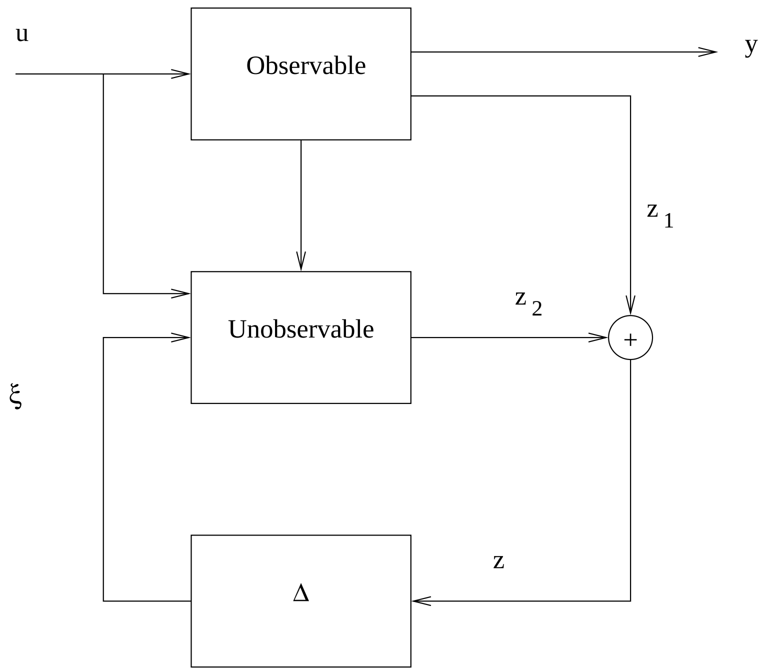
3. *For all $x_0 \in \mathbf{R}^n$, the supremum in $\sup_{\tau \geq 0} V_\tau(x_0)$ is achieved at $\tau = 0$.*

- This lemma is used to prove the following geometric characterisations of differential robust unobservability.

Theorem. *Suppose that $G(s) \equiv 0$. Then a state x_0 is differentially robustly unobservable if and only if it is an unobservable state for the pair (C_2, A).*

- The above theorem implies that when $G(s) \equiv 0$ the robustly unobservable set is a linear space equal to the unobservable subspace of the pair (C_2, A).

- From the above theorem and the fact that $G(s) \equiv 0$, it follows that we can apply the standard Kalman decomposition to represent the uncertain system as shown below.

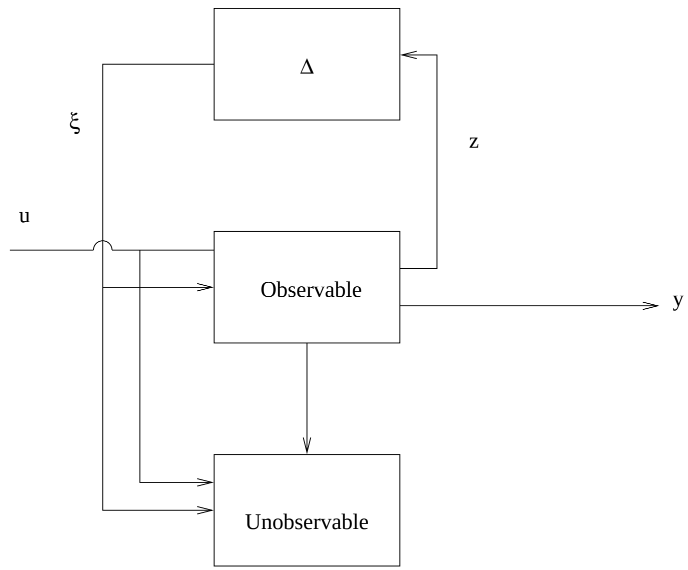


- Note that all of the uncertainty is in the unobservable subsystem.

We now consider the case in which $G(s) \not\equiv 0$.

Theorem. *Suppose that $G(s) \not\equiv 0$. Then a state x_0 is differentially robustly unobservable if and only if it is an unobservable state for the pair $(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, A)$.*

- The above theorem implies that when $G(s) \neq 0$, the robustly unobservable set is a linear space equal to the unobservable subspace of the pair $\left(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, A\right)$.
- From this theorem, it follows that we can apply the standard Kalman decomposition to represent the uncertain system as shown below:



- In this case, all of the uncertainty is in the observable subsystem or in the coupling between the two subsystems.

Geometric Results on Possible Controllabilty

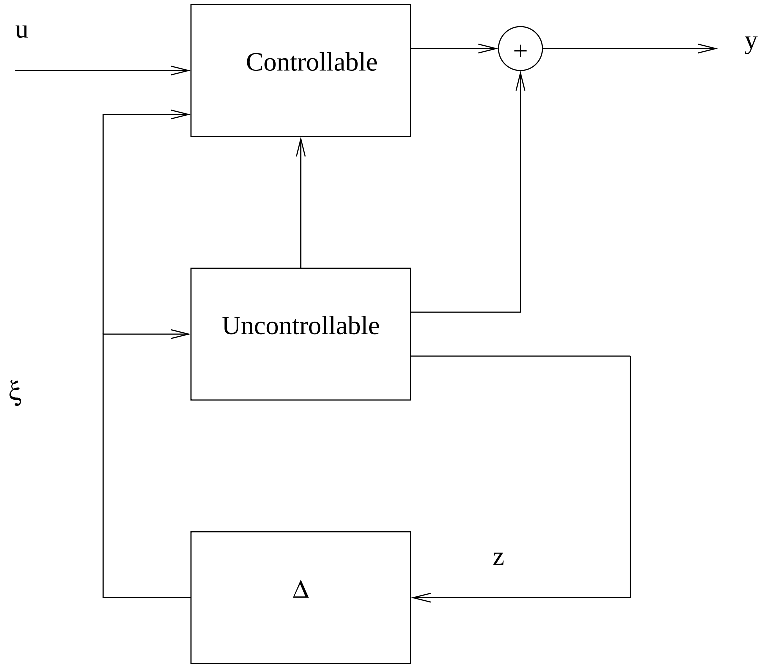
- The results in this case depend on the transfer function $H(s)$ to be the transfer function from the input $u(t)$ to the output $z(t)$; i.e.,

$$H(s) = C_1(sI - A)^{-1}B_1 + D_1.$$

> **Theorem.** *Suppose that $H(s) \equiv 0$. Then a state x_0 is differentially possibly controllable if and only if it is a controllable state for the pair (A, B_1).*

- The above theorem implies that when $H(s) \equiv 0$ the robustly unobservable set is a linear space equal to the unobservable subspace of the pair (A, B_1).

- From the above theorem and the fact that $H(s) \equiv 0$, it follows that we can apply the standard Kalman decomposition to represent the uncertain system as shown below.

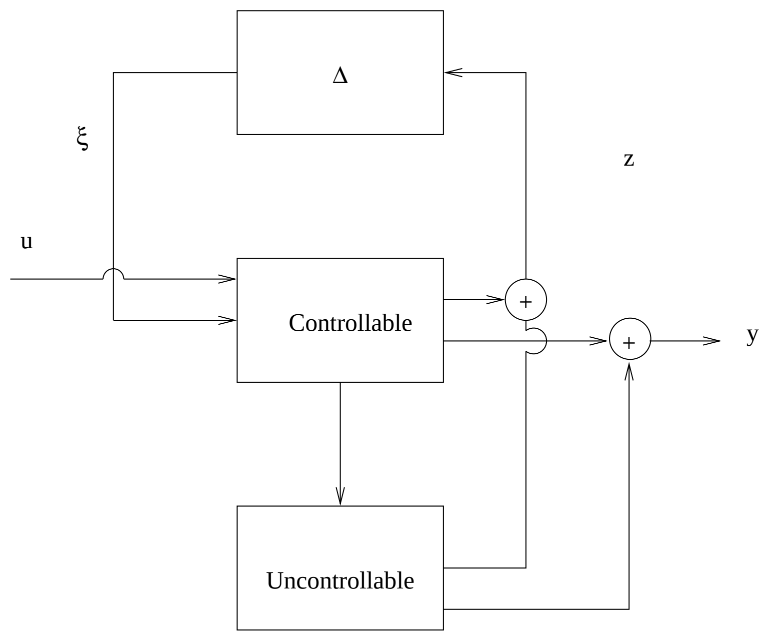


- In this case, we only have uncertainty in the uncontrollable subsystem.

- In the case that $H(s) \not\equiv 0$, we have only a partial result as follows:

> **Theorem.** *Suppose that $H(s) \not\equiv 0$. Then a state x_0 is differentially possibly controllable only if it is a controllable state for the pair $(A, [B_1\ B_2])$.*

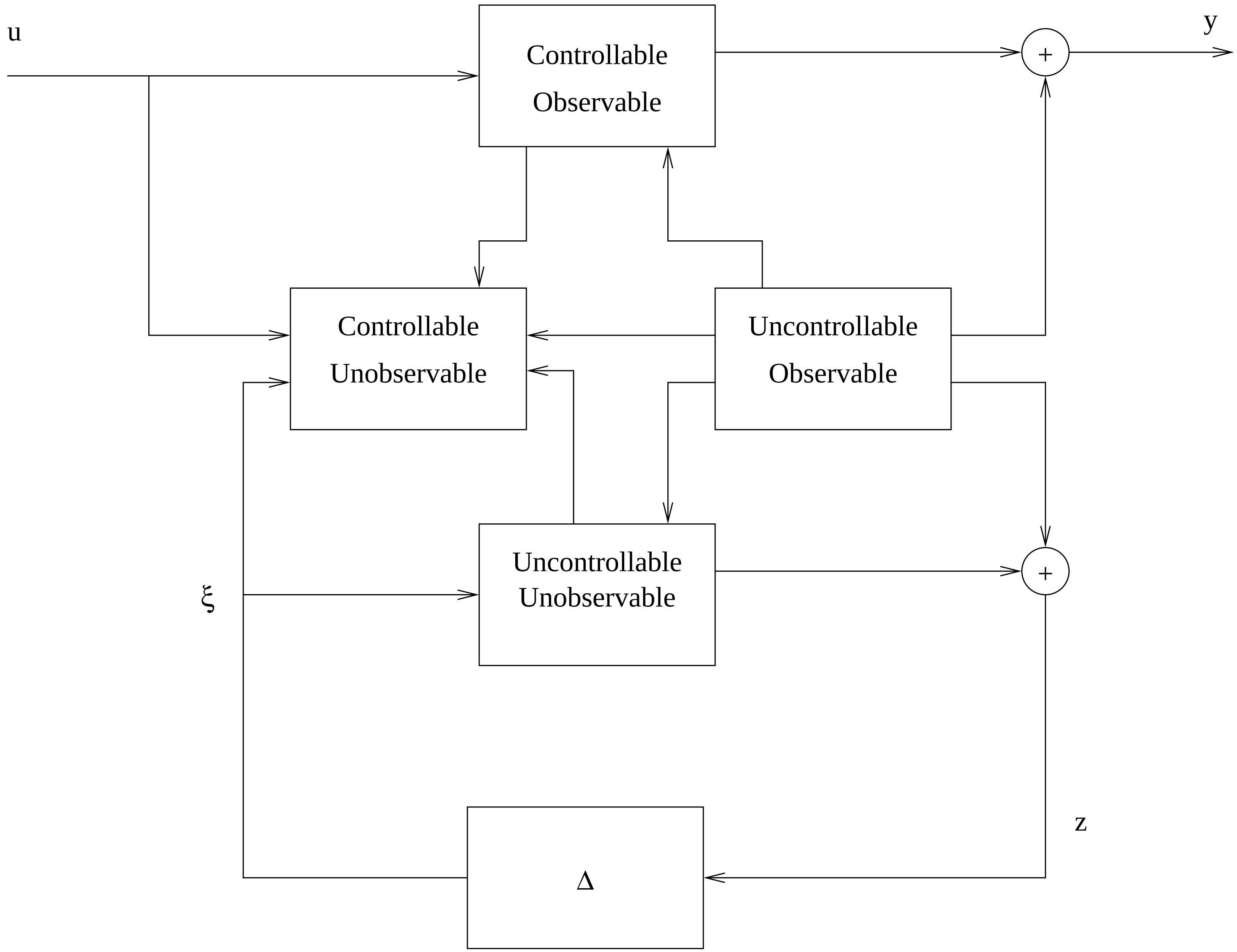
- To date we have been unable to prove the converse part of this theorem. If it were true, we would be able to represent the uncertain system as shown below in this case:



Kalman Decompositions

We can now combine our results to obtain a Kalman decomposition for the uncertain system at least in some cases:

Case 1 $G(s) \equiv 0$, $H(s) \equiv 0$. In this case, we would apply the standard Kalman decomposition to the triple (C_2, A, B_1) to obtain the situation as illustrated in the following block diagram.


u
y
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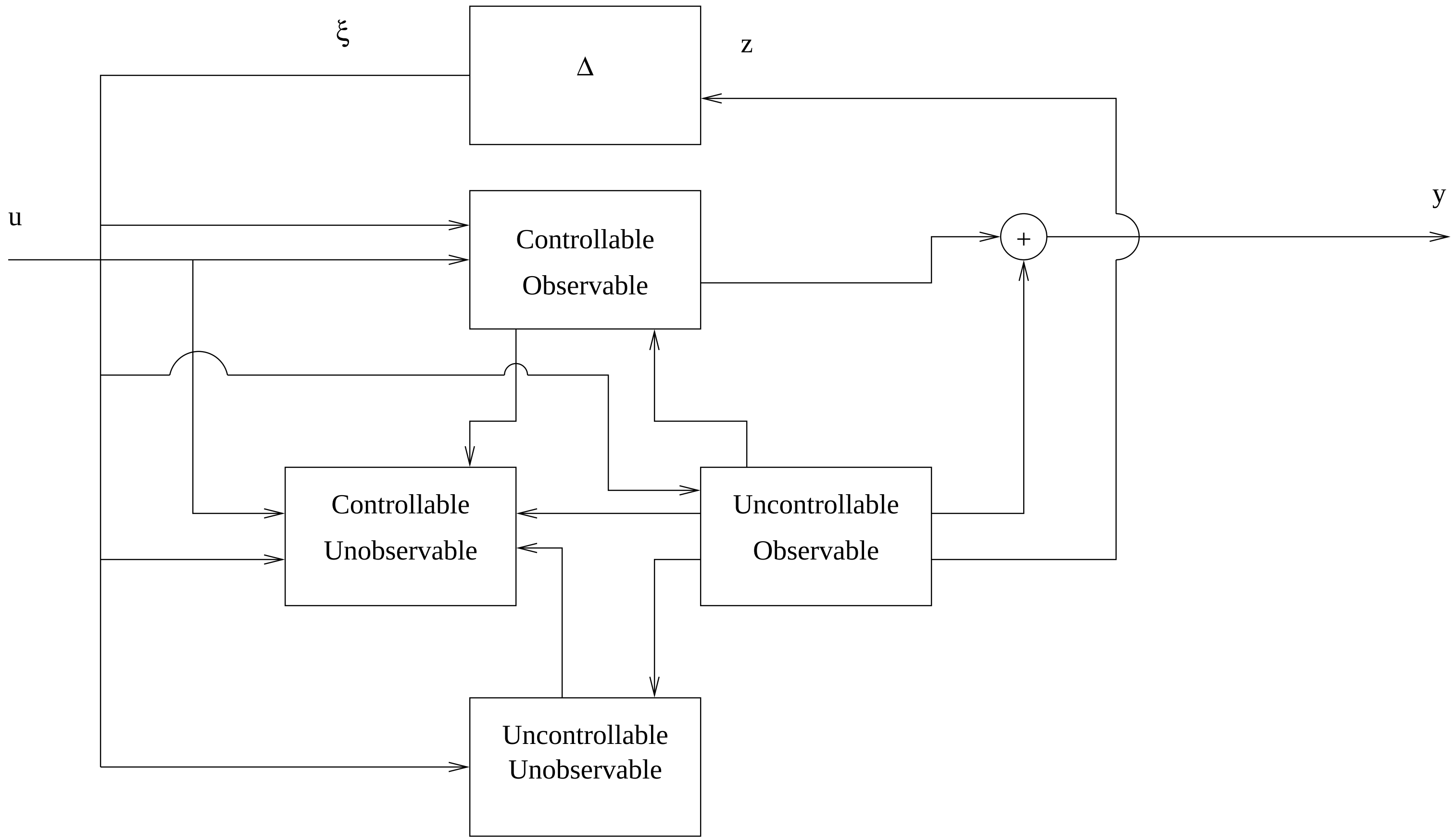
Case 2 $G(s) \not\equiv 0$, $H(s) \equiv 0$. In this case, we would apply the standard Kalman decomposition to the triple

$$\left(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, A, B_1 \right)$$

to obtain the situation as illustrated in the following block diagram.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Uncontrollable
Unobservable

- Note that in order to guarantee that the condition $H(s) \equiv 0$ we need to make a further restriction on the controllable observable block in the above diagram so that it in fact it only had an output y

Discussion

- Why do we consider differential versions of controllability and observability?
  - In the case of observability, it removes technical problems by ensuring that for at least one τ, the Q Riccati equation has a solution on $[0, T]$ and so $V_\tau(x_0, T) > -\infty$ for at least one value of τ.
  - In the case of controllability, it rules out counter examples of the form

$$\begin{aligned} \dot{x} &= ax + \xi \\ z &= x \end{aligned}$$

for which there exists a particular time varying uncertainty satisfying the IQC on $[0, T]$ and which drives any initial condition to zero at the particular time T. Such a system would be possibly controllable on $[0, T]$ and yet the control input u does not affect the system at all. Requiring T to be arbitrarily small rules out this.

- Note that the decompositions presented do not depend on the size of the uncertainty bound but only on the structure of the system.
- What happens if we considered constant norm bounded uncertainty rather than the IQC description considered?
  - It seems that in this case, the situation is much more complicated. For example

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ \delta & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ \delta \end{bmatrix} u$$

with $|\delta| \leq 1$ has a possibly controllable set which is a cone not a subspace; e.g. the controllability matrix is

$$\begin{bmatrix} 1 & 0 \\ \delta & 0 \end{bmatrix}$$

Future Research

- Resolve the possible controllability question for the case $H(s) \not\equiv 0$.
- Relate results to question of minimum realization for uncertain systems.
- Extend to the case of structured uncertainty.