On design of finite-level quantized feedback control

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Outline

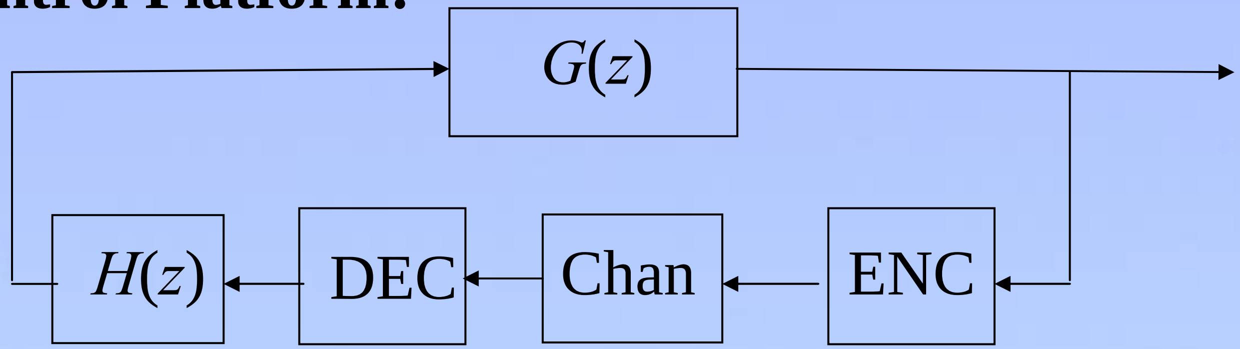
- Overview
- Control using Infinite-level Logarithmic Quantization
- Finite-level Quantized Feedback Stabilization
- Illustrative Examples
- Summary

Overview of Quantized Feedback Control

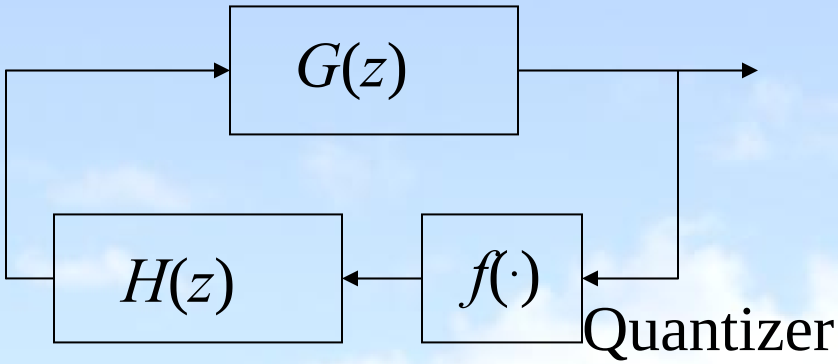
Brief History of Quantized Feedback Control

- Kalman ’56: Effect of quantization, limit cycles & their chaotic behaviours
- Mitigation of quantization effects: (Curry ’70, Delchamps ’90, Miller *et. al.* ’89)
- Absolute stability theory: treating quantization errors as nonlinearity (Desoer & Vidyasagar ’75, Willems ’71,…)
- Feedback using finite bandwidth communications links (Wong & Brockett ’99,’00 ,…)
- Minimal information feedback (Nair & Evans ’00, Elia & Mitter ’01, Elia et. al,’00,’01,’02, Kao & Venkatesh ’02,…)

Basic Control Platform:



By ignoring communications problems (e.g., transmission errors and delays), the design problem becomes a quantized feedback control problem:



For this talk, we limit to stabilization problems only.

Types of Quantizers

- **Static quantizers (or memoryless quantizers)**
  - **Logarithmic quantizers are optimal in many control settings (Elia & Mitter 2001, Fu and Xie 2004, …)**
  - **But requiring an infinite number of quantization levels**
- **Dynamic quantizers (with memory)**
  - **Recent focus has been on minimum information rate for feedback (Nair and Evans 2003,…).**
  - **Typically, a *very* low bit rate is needed**
  - **However, practical control design based on the minimum information rate is difficult.**

Example 1:

$$\begin{aligned} x_{k+1} &= a x_k + u_k \\ y_k &= x_k \end{aligned} \qquad (|a|>1)$$

Minimum bit rate required for stabilization = $\log_2|a|$.

For a = 8, it is only 3 bits.

Even for a = 128, it is only 7 bits!

In contrast, modern communication channels can handle data rates of KPS to MPS without any difficulties.

Key Question of the Paper:

How to design quantized feedback controllers which deliver good control performance, are easily implementable, yet, require only a moderate bit rate?

By moderate bit rate, we typically require several bits to several bytes per sample.

Infinite-Level Logarithmic Quantization

System:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned}$$

(SISO, unstable, minimal realization)

Static quantizer (to be designed):

$$v_k = Q(y_k)$$

where v_k takes values in the set

$$\mathcal{U} = \{\pm u_i : i = 0, \pm 1, \pm 2, \cdots\} \cup \{0\}$$

Feedback Controller (to be designed):

$$\begin{aligned} \hat{x}_{k+1} &= A_c\hat{x}_k + B_c v_k \\ u_k &= C_c\hat{x}_k + D_c v_k \end{aligned}$$

Notion of stability: Quadratic stability (i.e., fixed Lyapunov matrix)

Design Objective:

Global asymptotic stabilization (quadratic stabilization) with min. quantization density

Quantization density:
$$\eta_Q = \limsup_{\epsilon \longrightarrow 0} \frac{\#g[\epsilon]}{-\ln \epsilon}$$

where $\#g[\epsilon]$ is the number of quantization levels in $[\epsilon,\ 1/\epsilon]$

Known Results (Fu & Xie 2004):

- Optimal quantizer is logarithmic
- Quantization error for logarithmic quantizer = sector bound
- Solution to optimal quantized feedback stabilization can be solved via H_∞ optimization.

Details of the Known Results:

Logarithmic quantizer:

$$U = \{\pm u^{(i)} = \rho^i u^{(0)},\ i = 0, \pm 1, \pm 2, \ldots\} \cup \{0\}, \quad 0 < \rho < 1$$

$$Q(y) = \begin{cases} \rho^i u_0, & \text{if } \frac{1}{1+\delta}\rho^i u_0 < y \le \frac{1}{1-\delta}\rho^i u_0 \\ 0, & \text{if } y = 0 \\ -Q(-y), & \text{if } y < 0. \end{cases}$$

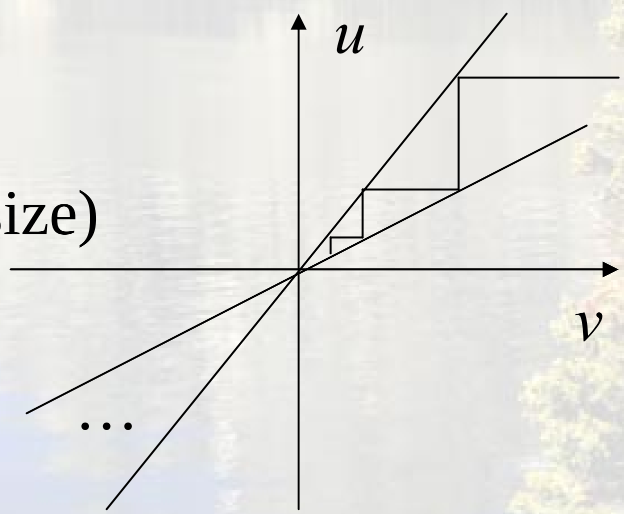
ρ represents quantization density

Associated sector bound:

$$\delta = \frac{1-\rho}{1+\rho} \text{ (sector size)}$$

Quantization error:

$$v = Q(y) = (1+\Delta)y, \quad |\Delta| \le \delta$$



Theorem (Fu & Xie 2004):

For a given logarithmic quantization density $\rho > 0$, the system is quadratically stabilizable via a quantized controller with quantization density ρ if and only if the following auxiliary system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ v_k &= (1+\Delta)Cx_k, \quad |\Delta| \le \delta \end{aligned}$$

is quadratically stabilizable. It follows that the largest sector bound δ_{sup} and the corresponding coarsest quantization density ρ_{inf} are given by

$$\begin{aligned} \rho_{\text{inf}} &= \frac{1-\delta_{\text{sup}}}{1+\delta_{\text{sup}}} \\ \delta_{\text{sup}}^{-1} &= \inf_{H(z)} ||(1-H(z)G(z))^{-1}H(z)G(z)||_\infty \end{aligned}$$

A SPECIAL CASE:
SISO Stabilization using Quantized State Feedback

(Revisit of Elia and Mitter '01)

System: $$x[k+1] = Ax[k] + Bu[k]$$

Controller: $$u[k] = f(v[k])$$

$$v[k] = Kx[k]$$

where $f(\cdot)$ is a logarithmic quantizer with density ρ.

Quadratic stabilization is considered.

Lyapunov function: $$V(x) = x^T P x$$

Stabilization criterion:

$$\nabla V(x) = V(Ax + Bf(Kx)) - V(x) < 0, \ \forall x \neq 0$$

Theorem (Elia & Mitter '01):

$$\rho_{\min} = \frac{1-\delta}{1+\delta}, \qquad \delta^{-1} = \prod_i |\lambda_i^u|$$

Alternative proof by sector bound method:

Minimizing the H_∞ norm of the following function:

$$G_C(z) = K(zI - A - BK)^{-1} B$$

More specifically, the coarsest quantization density is determined by

$$\sup_K \delta = \frac{1}{\inf_K \| G_C(z) \|_\infty}$$

Solving the above gives

Example:

$$G(z) = \frac{z-3}{z(z-2)}$$

Using quantized state feedback,

$$\delta^{-1} = 2 \;\; \Rightarrow \;\; \rho_{\min} = \frac{1-1/2}{1+1/2} = 0.3333$$

Using output feedback with quantized output measurements,

$$\delta^{-1} = 10 \;\; \Rightarrow \;\; \rho_{\min} = \frac{1-1/10}{1+1/10} = 0.8182$$

Another Feedback Configuration for Quantized Output Feedback

System:

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k]$$

controllable & observable

Control signal is quantized but the output measurement is not.

Theorem: For quantized control input, the coarsest quantization density for state feedback stabilization is also achievable by output feedback. In particular, the required controller is an observer-based one:

$$x_c[k+1] = Ax_c[k] + Bu[k] + L(y[k] - Cx_c[k])$$
$$v[k] = Kx_c[k]$$
$$u[k] = f(v[k])$$

Proof (key idea): Design a deadbeat observer. This gives perfect state observation in a finite number of steps. Afterwards, output feedback is the same as state feedback.

Extension to MIMO Systems

System:

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k]$$

Controllable & observable

Same as before except m-inputs, m-outputs

Quantizer:

$$f(v) = \text{diag}\{f_1(v_1), f_2(v_2), \cdots, f_m(v_m)\}$$

Quantization density vector:

$$\rho = [\rho_1, \rho_2, \cdots, \rho_m], \; 0 < \rho_i < 1$$

Quantization error:

$$e(v) = \Delta(v)v,$$
$$\Delta(v) = \text{diag}\{\Delta_1(v_1), \Delta_2(v_2), \cdots, \Delta_m(v_m)\}$$
$$|\Delta_i(v_i)| \le \delta_i, \quad i = 1, 2, \cdots, m$$

Quantized State Feedback or Control Input

Theorem: For a given quantization density vector ρ, consider the following auxiliary system:

$$x[k+1] = Ax[k] + B(I+\Delta)v[k]$$

$$\Delta = \mathrm{diag}\{\Delta_1, \Delta_2, \cdots, \Delta_m\}, \quad |\Delta_i| \le \delta_i$$

Suppose the auxiliary system is quadratically stabilizable, then the original system is quadratically stabilizable via quantized feedback.

Conversely, if the original system is quadratically stabilizable via quantized feedback, and in addition, $\ln \rho_i / \ln \rho_j$ are irrational $\forall i \neq j$, and δ_i are replaced with $\delta_i - \varepsilon$ for arbitrarily small $\varepsilon > 0$.

Then, the auxiliary system is quadratically stabilizable.

The two problems are “equivalent”

Finite-Level Logarithmic Quantization

Motivation: A logarithmic quantizer has an infinite number of quantization levels.

Objectives:

- To achieve global asymptotic stabilization using a finite-level quantizer
- To show that the information bit rate is very moderate

Our Approach:

- Using a finite-level (static) logarithmic quantizer, plus
- Dynamic scaling

Finite-level logarithmic quantizer:

$$\mathcal{U} = \{\pm\rho^i u_0, i = 0, 1, 2, \cdots, N-1\}, \quad u_0 > 0$$

$$Q(y) = \begin{cases} \rho^i u_0, & \text{if } \frac{1}{1+\delta}\rho^i u_0 < y \leq \frac{1}{1-\delta}\rho^i u_0, \\ & \quad 0 \leq i < N \\ \rho^{N-1} u_0, & \text{if } \ 0 \leq y \leq \frac{1}{1+\delta}\rho^{N-1} u_0, \\ u_0, & \text{if } \ y > \frac{1}{1-\delta}\rho^i u_0, \\ -Q(-y), & \text{if } y < 0. \end{cases}$$

(There are 2N levels)

Dynamic Scaling:

scaling factor

$$v_k = g_k^{-1} Q(g_k y_k)$$

with

$$g_{k+1} = \begin{cases} g_k \gamma_1, & \text{if } |Q(g_k y_k)| = u_0 \\ g_k / \gamma_2, & \text{if } |Q(g_k y_k)| = \rho^{N-1} u_0 \\ g_k, & \text{otherwise} \end{cases}$$

where $0 < \gamma_1, \gamma_2 < 1$ are design parameters.

Control Design:

Step 1: Design a stabilizing quantized feedback controller $H(z)$ using an infinite-level logarithmic quantizer with quantization density $0 < \rho < \rho_{\text{inf}}$. This controller will be used with a finite-level quantizer and a dynamic scaling factor.

Step 2: Choose $0 < \gamma_1, \gamma_2 < 1$ for dynamic scaling. Roughly speaking,

γ_1 is chosen so that $x(k+1) = \gamma_1 A\, x(k)$ is stable;

γ_2 is chosen so that $H(z)$ still stabilizes

$$\gamma_2 x(k+1) = Ax + Bu$$

$$v(k) = (1+\Delta)Cx, \ |\Delta| \le \delta$$

Step 3: Choose $N > N_0$ which depends on $\gamma_1, \gamma_2, H(z)$

Main Result:

Let the controller $H(z)$, scaling parameters γ_1 and γ_2, and the number of quantization levels N be designed as above. Then, the closed-loop system is globally asymptotically stable.

Typical Behavior of the System:

If the initial state is very large, the feedback signal tends to be saturated, forcing g_k to decrease fast. This would result in a period of overshoot. Once g_k is sufficiently small, saturation will stop and the state decays exponentially. When the state is sufficiently small, g_k will increase gradually, causing the quantizer to bounce back and forth between the dead zone and logarithmic region. During this phase, the state also decays exponentially, but at a lower rate.

Remark 1:

The main advantage of the proposed scheme is that the system behaves as if there were an infinite number of logarithmic quantization levels when the initial state is “moderate” in size, i.e., the state would converge exponentially. Only when the initial state is very large, a transient overshoot can be present.

The region of exponential convergence can be easily increased by using more quantization levels, and the number of feedback information bits grows only at a log(log(·)) rate when the size of this region increases.
This makes the proposed scheme very practical.

Reason for log(log(.)): N = log (state size) due to logarithmic quantizer; bit # = $\log_2 (2N)$.

Remark 2 (on signal transmission):

Only the quantized information v_k needs to be transmitted. The receiver can automatically detect the dynamic scaling factor g_k based on the following assumptions:

- Both sides of the feedback channel knows the values of g_0, γ_1 and γ_2.
- There are no transmission errors or delays.

Simple Detection algorithm:

- If v_k is saturated, $g_{k+1} = g_k\gamma_1$;
- If v_k is in the deadzone, $g_{k+1} = g_k/\gamma_2$;
- Otherwise (v_k is in the logarithmic region), $g_{k+1} = g_k$.

Minimum Number of Quantization Levels:

$$N_0 = 1 + \frac{2\log(\gamma_2 - \sqrt{1-\eta}) - \log(\bar{B}^T P \bar{B} \bar{C} P^{-1} \bar{C}^T)}{2\log(\rho)}$$

Design parameters (scalars)

Quantization density

Closed-loop system matrices (involving controller)

Lyapunov matrix for closed-loop system

Numerical search

Given

Optimized using LMIs

Robustness Against Additive Noises

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k \\ y_k &= Cx_k, \end{aligned}$$

$$||w_k|| \leq \bar{w}$$

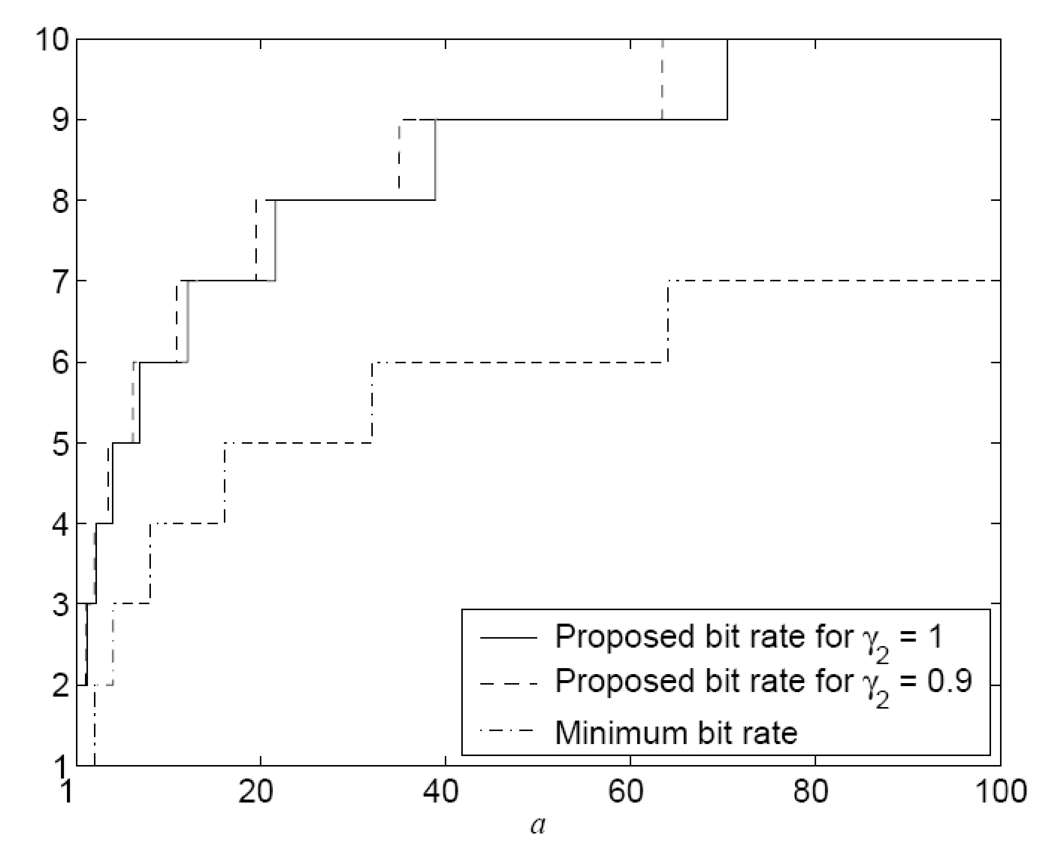
Main Result: Suppose the quantized feedback controller and the dynamic scaling factor are designed as in the noise-free case, and the scaling factor is saturated at some $\bar{g}$.
Then, the state x_k will converge to a bounded region asymptotically, and the size of this asymptotic region is proportional to $\bar{w}$.

The upper bound for g can be optimized using LMIs.

Illustrative Examples

Example 1: Recall the scalar system

$$x_{k+1} = ax_k + u_k \quad (|a|>1)$$



Note: It is very impractical from control point of view to use a large *a*.

Example 2:

$$A = \begin{bmatrix} 2.7 & -2.41 & 0.507 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

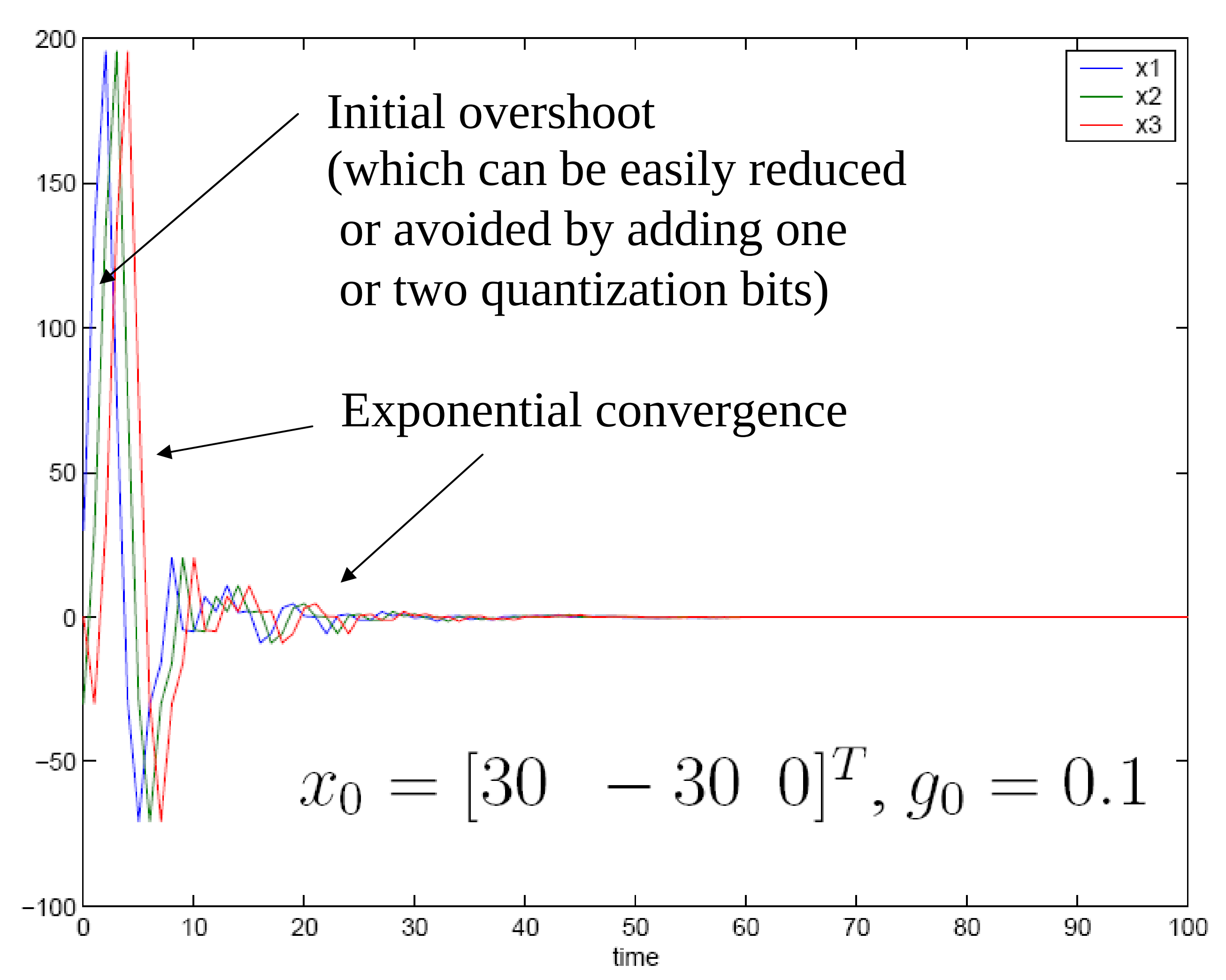
$$C = [1 \quad -0.5 \quad 0.04]$$

two unstable open-loop poles at $1.2 \pm i0.5$

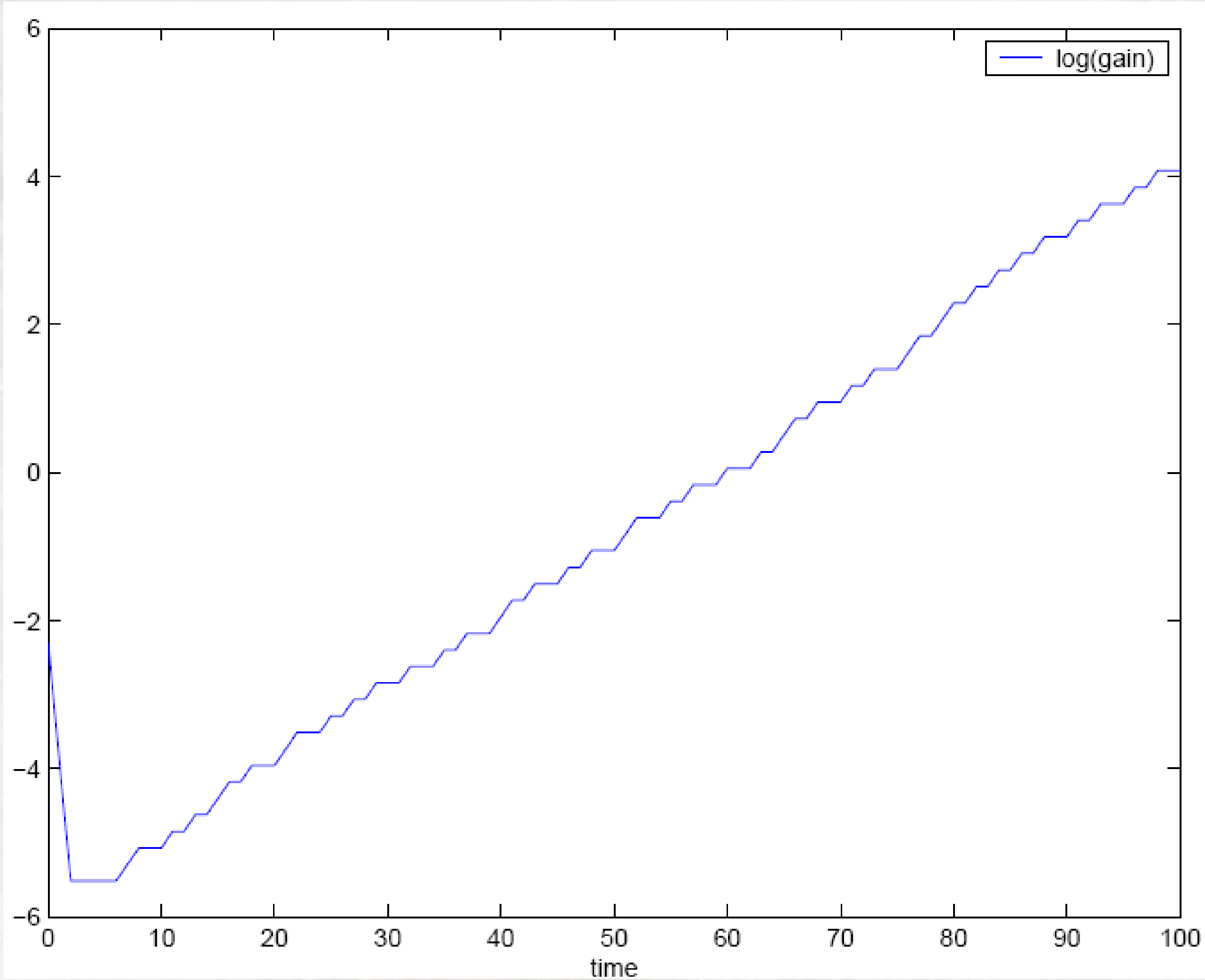
Minimum bit rate for stabilization = 1 bit.

Our design leads to a 4-bit logarithmic quantizer.

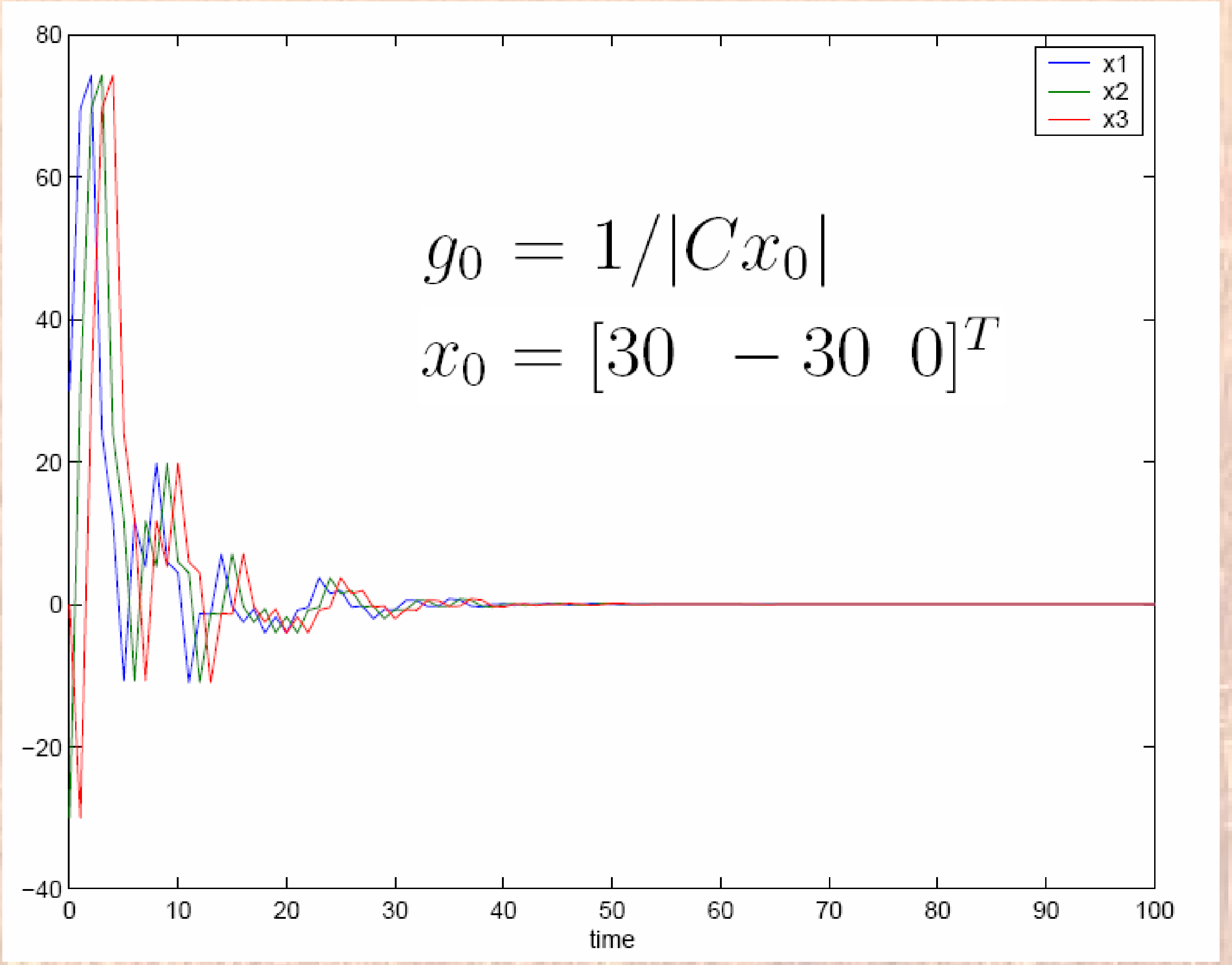
Sample State Trajectory (Initial state unknown):



Dynamic Scaling Factor:

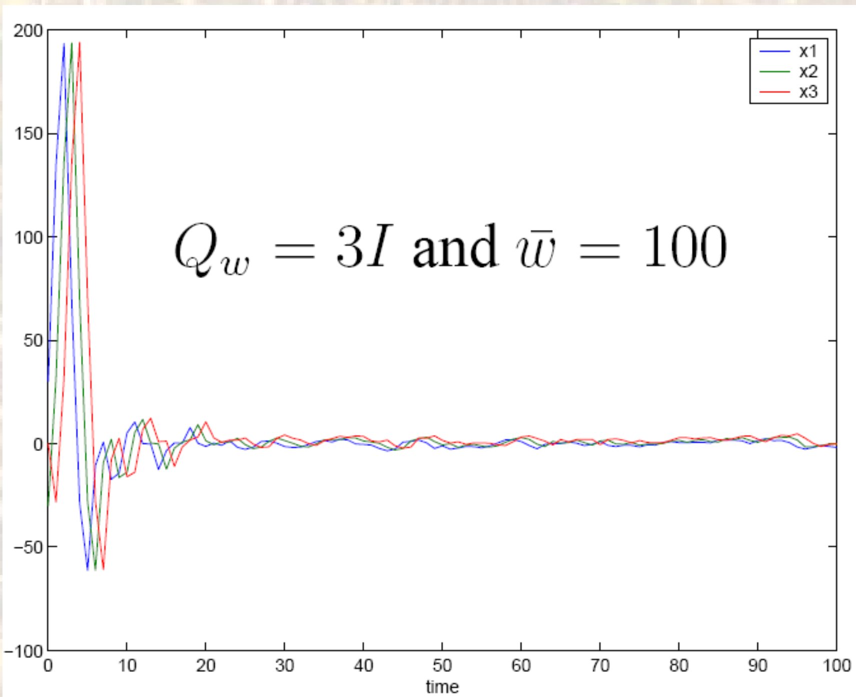


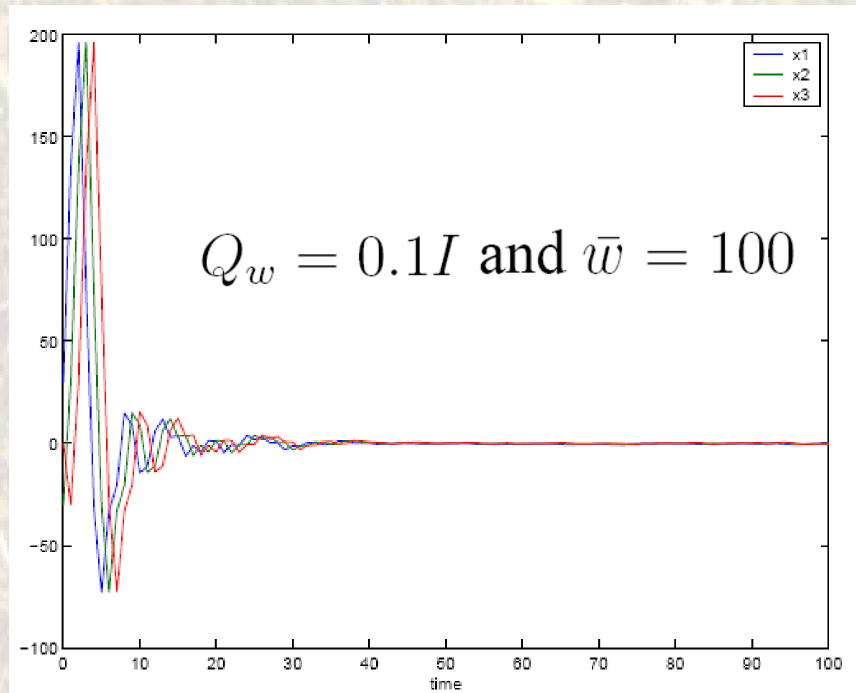
Sample State Trajectory (Initial state known):



Robustness against additive noises:

$$x_0 = [30 \quad -30 \quad 0]^T$$





Conclusions

We have proposed a simple dynamic scaling method for quantized feedback control. This allows us to achieve asymptotic stabilization using a very moderate number of quantization levels. The proposed control scheme can be easily implemented, has nice convergence properties, and robust against measurement noises.

New challenges:

- Packet dropouts
- Non-detectable transmission errors
- Transmission delays