Recent results in data-driven control

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(joint work with Ivan Markovsky)

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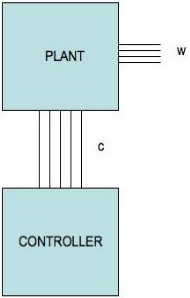
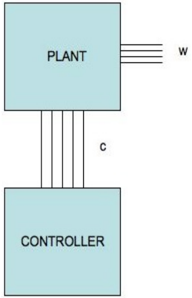
Outline

Model-based control



Model-based control

$$\frac{d}{dt}x = Ax + Bu$$
$$y = Cx + Du$$



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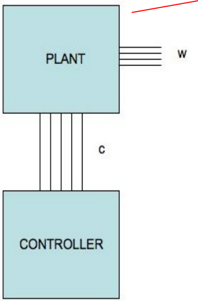

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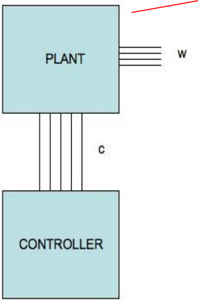

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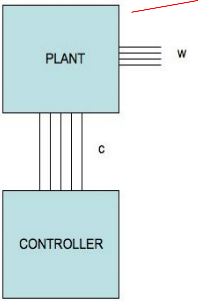

Performance criterion (quadratic, H_∞, etc.)

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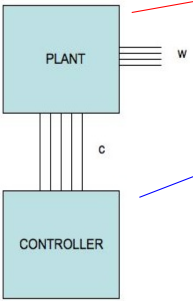

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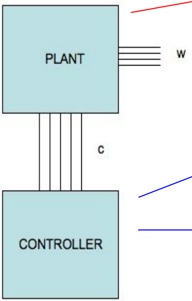

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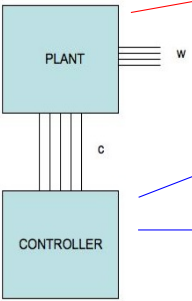

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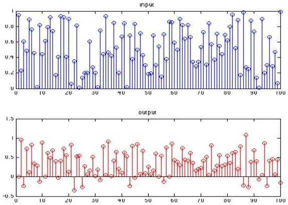
$$\frac{d}{dt}x' = A'x' + B'u'$$
$$y' = C'x' + D'u'$$

Performance criterion (quadratic, H_∞, etc.)

Data-based control approaches

Open-loop plant trajectory + Performance criterion $\Longrightarrow$ **Controller representation or control input**



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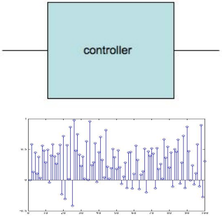

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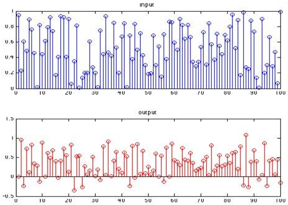

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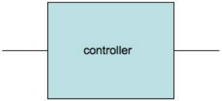

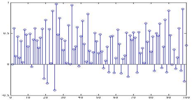

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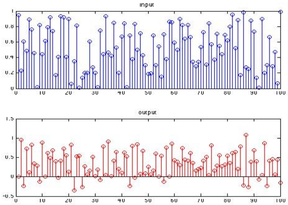

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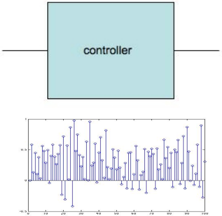

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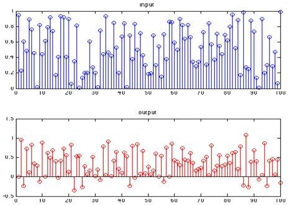

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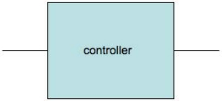

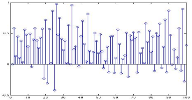

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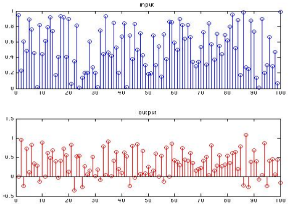

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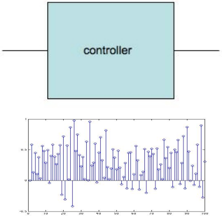

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- **Off-line**

Data-based control approaches

- **J.-T.H. Chan: *LQ*-controller;**
- **K. Furuta and M. Wongsaisuwan: *LQG*-controller from Markov parameters;**
- **Y. Kawamura: *LQ* output dynamic controller via orthogonalization of signals;**
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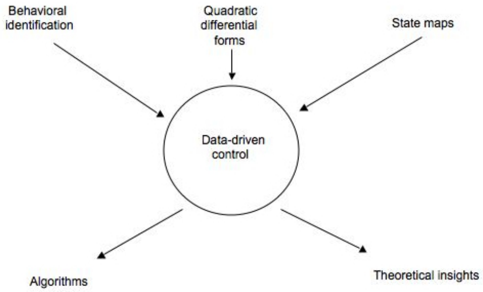

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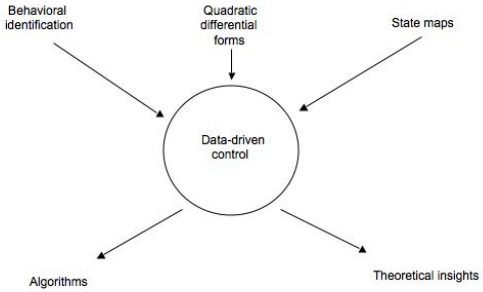

- **Output: control signals, not controller representation;**

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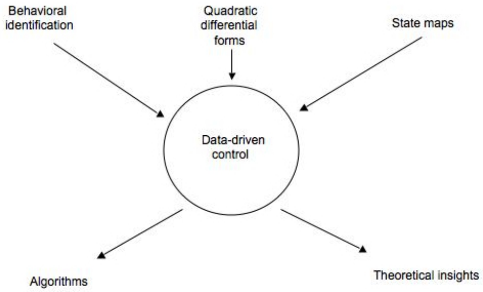

- **Output: control signals, not controller representation;**

- ***A priori* representation of plant not assumed: "*let the data speak*";**

The data-driven approach

Done:

- ***LQ*-tracking**
- **Algorithms based on "data-driven simulation"**

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- ***LQ*-tracking**
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In this talk:

- **Quadratic finite-horizon control;**
- **Exact data from a LTI system;**
- **Solution;**
- **Why “state” feedback?**
- **Orthogonality property.**

Outline

Introduction

Prolegomena and problem statement

Solution of data-driven LQ finite-horizon control problem

The orthogonality property

Why “state” feedback?

- An intrinsic definition of “state”
- Optimal ***LQ***-cost is a function of the state

Discrete-time behaviors

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Discrete-time behaviors

- **LTI system $\Sigma \rightsquigarrow$ trajectories *w*, with $\mathtt{w}$ components**
- **Set of all trajectories of Σ is behavior $\mathfrak{B}_\Sigma$**
- **$\mathfrak{B}_\Sigma$ representable with state-space, difference equations of various types, transfer function**
- **Integer structural invariants of $\mathfrak{B}_\Sigma$:**
  - **order $\mathtt{n}(\mathfrak{B}_\Sigma)$= minimal dimension of state space**
  - **lag $\mathtt{L}(\mathfrak{B}_\Sigma)$=observability index in any minimal state-space representation of $\mathfrak{B}_\Sigma$**

Identifiability and persistency of excitation

¿Given $\widehat{w}(1), \dots, \widehat{w}(T) \in \mathfrak{B}|_{[1,T]}$ and $1 \leq L \leq T$, when is

$\mathfrak{B}|_{[1,L]} = \text{span}\left\{\widehat{w}(k), \dots, \widehat{w}(k+L) \mid k = 1, \dots, T-L\right\}$?

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¿When is (a window of) the data
a "faithful" representation of the behavior ?

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$$\text{im}\underbrace{\left(\begin{bmatrix} \widehat{w}(1) & \widehat{w}(2) & \cdots & \widehat{w}(L) \\ \widehat{w}(2) & \widehat{w}(3) & \cdots & \widehat{w}(L+1) \\ \vdots & \vdots & \cdots & \vdots \\ \widehat{w}(T-L) & \widehat{w}(T-L+1) & \cdots & \widehat{w}(T) \end{bmatrix}\right)}_{\mathcal{H}_{T-L}(\widehat{w})} \rightsquigarrow \mathfrak{B}|_{[1,L]}$$

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True if

$$\text{rank}\left(\begin{bmatrix} \widehat{u}(1) & \widehat{u}(2) & \cdots & \widehat{u}(L) \\ \widehat{u}(2) & \widehat{u}(3) & \cdots & \widehat{u}(L+1) \\ \vdots & \vdots & \cdots & \vdots \\ \widehat{u}(T-L) & \widehat{u}(T-L+1) & \cdots & \widehat{u}(T) \end{bmatrix}\right) = (T-L)\mathtt{u}+\mathtt{n}$$

Input component $\widehat{u}$ of $\widehat{w} = (\widehat{u}, \widehat{y})$
is persistently exciting (PE) of order T-L+$\mathtt{n}$

(Fundamental Lemma)

Finite-horizon data-driven control problem

Given:

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2. $T_f \geq \mathtt{L}(\mathfrak{B})$;
3. $\Phi = \Phi^\top > 0 \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$;
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open-loop data of plant

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"initial condition"

horizon

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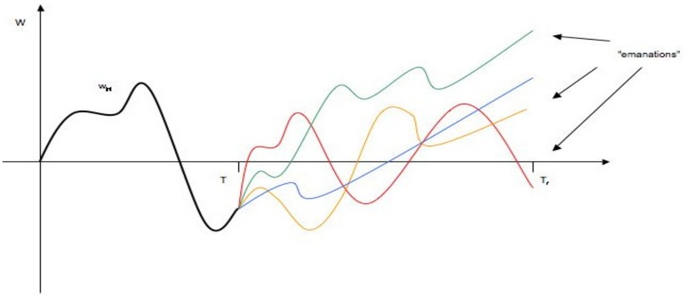

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$$w' := w_{\mathrm{ini}} \underset{T}{\wedge} w^\star := \begin{cases} w_{\mathrm{ini}}(k) & \text{for } k \leq T \\ w^*(k) & \text{for } T < k \leq T_f \end{cases}$$

minimizes

$$\sum_{t=1}^{T_f} w'^\top(t)\Phi w'(t)$$

subject to $w' \in \mathfrak{B}|_{[1,T_f]}$

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concatenation at T

Outline

Step 1: generate all emanations

Basic idea: use Hankel matrix of data

Under Fundamental Lemma assumptions,
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**Under Fundamental Lemma assumptions,
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Free response emanating from w_{ini}:

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$\bar{w}_0$ is unique

Step 1: generate all emanations

Basic idea: use Hankel matrix of data

Under Fundamental Lemma assumptions,
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Responses with zero-past until T:

$$\begin{bmatrix} \widehat{w}(1) & \widehat{w}(2) & \cdots & \widehat{w}(T') \\ \widehat{w}(2) & \widehat{w}(3) & \cdots & \widehat{w}(T'+1) \\ \vdots & \vdots & \cdots & \vdots \\ \widehat{w}(T) & \widehat{w}(T+1) & \cdots & \widehat{w}(T'+T-1) \\ \widehat{w}(T+1) & \widehat{w}(T+1) & \cdots & \widehat{w}(N-1) \\ \vdots & \vdots & \cdots & \vdots \\ \widehat{w}(T_f) & \widehat{w}(T_f+1) & \cdots & \widehat{w}(N) \end{bmatrix} \gamma = \underbrace{\begin{bmatrix} 0_{\mathtt{w}T} \\ w(T+1) \\ \vdots \\ w(T_f) \end{bmatrix}}_{\in \mathfrak{B}_Z}$$

zero past responses

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$\mathcal{H}_Z$ **full column matrix generating**

$$\mathfrak{B}_Z := \{w = (0_{wT}, w(T+1), \ldots, w(T_f)) \in \mathfrak{B}_{|[1,T_f]}\}$$

space of responses with zero past until T

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fixed ($\overline{w}_0$) **free** (β)

Concatenation no problem

Step 2: minimize

$\tilde{\Phi} \in \mathbb{R}^{\mathtt{w}T_f \times \mathtt{w}T_f}$ **block-diagonal from Φ**

$$\min \sum_{t=1}^{T_f} w'^\top(t) \Phi w'(t) \quad \text{s.t. } w' = w_{\text{ini}} \wedge_T w \in \mathfrak{B}|_{[1,T_f]} \quad \rightsquigarrow \quad \min_\beta \left(\overline{w}_0 + \mathcal{H}_Z \beta\right)^\top \tilde{\Phi} \left(\overline{w}_0 + \mathcal{H}_Z \beta\right)$$

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Minimum for β^* **such that**

$$\mathcal{H}_Z^\top \tilde{\Phi} \overline{w}_0 + \mathcal{H}_Z^\top \tilde{\Phi} \mathcal{H}_Z \beta^* = 0$$

Optimal trajectory is

$$w^* := \left[I - \mathcal{H}_Z \left(\mathcal{H}_Z^\top \tilde{\Phi} \mathcal{H}_Z \right)^{-1} \mathcal{H}_Z^\top \tilde{\Phi} \right] \overline{w}_0$$

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Numerical linear algebra-based algorithms for computing $\mathcal{H}_Z$ and $\mathcal{H}_F$

Outline

The orthogonality property

Define $\langle w_1, w_2 \rangle_{\tilde{\Phi}} := \sum_{k=1}^{T_f} w_1^\top(k) \Phi w_2(k)$ **and**

$$\mathfrak{B}^{\perp_\Phi}_{|[1,T_f]} := \{ w_{|[1,T_f]} \mid \langle w, v \rangle_\Phi = 0 \text{ for all } v \in \mathfrak{B} \}$$

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Of course, consequence of least-squares setting

Orthogonality property for state-space

Theorem: $\mathfrak{B} := \{(x, u) \mid \sigma x = Ax + Bu\}$ with cost

$$\sum_{k=1}^{T_f} \left[x^\top(k) Q x(k) + u^\top(k) R u(k)\right]$$

($Q > 0$, $R > 0$). Then

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Also, $(x^*, u^*) \perp_{\tilde{\Phi}} \mathfrak{B}_Z$ iff

$$u^*(i) = -(R + B^\top K_{i+1} B)^{-1} B^\top K_{i+1} A x^*(i)$$

where

$$\begin{aligned} K_{T_f} &:= Q; \\ K_i &= A^\top K_{i+1} A + Q \\ &- A^\top K_{i+1} B (R + B^\top K_{i+1} B)^{-1} B^\top K_{i+1} A \end{aligned}$$

Outline

Introduction

Prolegomena and problem statement

Solution of data-driven LQ finite-horizon control problem

The orthogonality property

Why “state” feedback?

- An intrinsic definition of “state”
- Optimal ***LQ***-cost is a function of the state

Conventional wisdom revisited

"In an LQ-problem the value of the optimal trajectory at time k is a linear function of the state at time k."

This truth can be deduced from first principles.

It is not only a consequence of the use of state-space representations!

State: the basic idea

Semifinal of the World Cup. You're late...

119 minutes late: 1-0

121 minutes late: 2-0

The current score is what matters...

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The current score is what matters...

- **The state contains all the relevant information about the future behavior of the system**
- **The state is the memory of the system**
- **Independence of past and future given the state**

The axiom of state

Shift-invariant behavior with latent variable x

$$\mathfrak{B}_{\text{full}} = \{(w, x)\}$$

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Shift-invariant behavior with latent variable x

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$\mathfrak{B}_{\text{full}}$ **is a *state system* if**

$$(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}} \text{ and } x_1(0) = x_2(0)$$

$$\Downarrow$$

$$(w_1, x_1) \underset{0}{\wedge} (w_2, x_2) \in \mathfrak{B}_{\text{full}}$$

Graphically...

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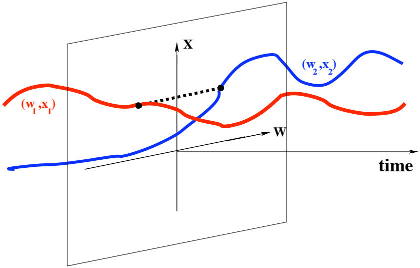

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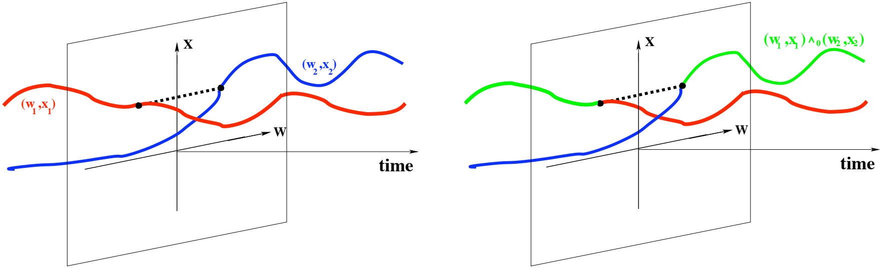

The axiom of state revisited

A *linear* $\mathfrak{B}_{\text{full}}$ with latent variable *x* is a state system if

$$(w, x) \in \mathfrak{B}_{\text{full}} \text{ and } x(0) = 0$$
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- **Concatenability with zero trajectory** is key.

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- **Concatenability with zero trajectory is key.**
- **Let us study this for $\mathfrak{B}$ described by**

$$R_0 w + R_1 \sigma w + \ldots + R_L \sigma^L w = 0$$

When is $\mathbf{w} \in \mathfrak{B}$ concatenable with zero?

$$R_0 w + R_1 \sigma w + \ldots + R_L \sigma^L w = 0$$

...	0	0	$w(0)$	$w(1)$	$w(2)$	$w(3)$	...
...	$k = -2$	$k = -1$	$k = 0$	$k = 1$	$k = 2$	$k = 3$	...

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State maps

$$\begin{aligned} R_1 w(0) + \cdots + R_L w(L-1) &= 0 \\ R_2 w(0) + \cdots + R_L w(L-2) &= 0 \\ \vdots \quad &= \vdots \\ R_L w(0) &= 0 \end{aligned} \quad \Longleftrightarrow \quad (X(\sigma) w)(0) = 0$$

where

$$X(\xi) := \begin{bmatrix} R_1 + \cdots + R_L \xi^{L-1} \\ R_2 + \cdots + R_L \sigma^{L-2} \\ \vdots \\ R_L \end{bmatrix}$$

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State map $X(\sigma)$ induces state variable.

Algorithms to compute minimal ones.

Optimal cost is quadratic function of the state

***Proposition*: Denote with $V(i)$ the minimum value of**

$$\min \frac{1}{2} \sum_{j=i}^{T_f} w(j)^\top \Phi w(j)$$

$$\text{subject to } w_{|_{[i,Tf]}} \in \mathfrak{B}_{|_{[i,Tf]}}$$

$$w(j) = \overline{w}_j \text{ given, } j = i, \ldots, i + \mathtt{L}(\mathfrak{B}) - 1$$

Let w^* be such that $V(i) = \frac{1}{2} \sum_{j=i}^{T_f} w^*(j)^\top \Phi w^*(j)$.

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Then for every state map $X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ there exists $K_i > 0$ in $\mathbb{R}^{\bullet \times \bullet}$ such that

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There is no "state" in problem statement!

Optimal cost is quadratic function of the state

Proof: **Take "minimal" kernel representation $R(\sigma)w = 0$; then lag L of $R = \mathtt{L}(\mathfrak{B})$. "Slide" $(R_0, R_1, \ldots, R_L)$ on**

$$\underbrace{\overline{w}(i), \ldots, \overline{w}(i+L-1)}_{=:\overline{W}}, \underbrace{w(i+L), \ldots, w(i+2L-1)}_{=:W_1}, \underbrace{w(i+2L), \ldots, w(T_f)}_{=:W_2}$$

and obtain conditions

$$\begin{bmatrix} R_0 & \cdots & \cdots & R_{L-1} \\ 0 & R_0 & \cdots & R_{L-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & R_0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \overline{W} = - \begin{bmatrix} R_L & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & & & \vdots \\ R_1 & \ddots & R_L & \ddots & & & \vdots \\ R_0 & R_1 & \ddots & R_L & \ddots & & \vdots \\ 0 & R_0 & \ddots & \ddots & R_L & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & R_0 & R_1 & \cdots & R_L \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$$

equivalent to constraints in optimization problem.

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Standard problem, with equality constraints:

$$\begin{bmatrix} \bar{x} \\ 0 \end{bmatrix} = - \underbrace{\begin{bmatrix} R_L & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & & & \vdots \\ R_1 & \ddots & R_L & 0 & & & \vdots \\ R_0 & R_1 & \ddots & R_L & \ddots & & \vdots \\ 0 & R_0 & R_1 & \ddots & R_L & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & R_0 & R_1 & \cdots & R_L \end{bmatrix}}_{=: A_i} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$$

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For simplicity A_i full row rank; optimum achieved for

$$\begin{bmatrix} w^*(i+L) \\ \vdots \\ w^*(T_f) \end{bmatrix} = \tilde{\Phi}^{-1} A_i^\top \left(A_i \tilde{\Phi}^{-1} A_i^\top \right)^{-1} \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix}$$

Optimal cost is quadratic function of the state

The cost of

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defining K_i as $(1,1)$-block of $\left(A_i \tilde{\Phi}^{-1} A_i^\top \right)^{-1}$:

$$V(i) = \begin{bmatrix} \overline{x} \\ 0 \end{bmatrix}^\top \left(A_i \tilde{\Phi}^{-1} A_i^\top \right)^{-1} \begin{bmatrix} \overline{x} \\ 0 \end{bmatrix} = \overline{x}^\top K_i \overline{x}$$

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$$\begin{bmatrix} w^*(i+L) \\ \vdots \\ w^*(T_f) \end{bmatrix} = \tilde{\Phi}^{-1} A_i^\top \left(A_i \tilde{\Phi}^{-1} A_i^\top \right)^{-1} \begin{bmatrix} \overline{x} \\ 0 \end{bmatrix}$$

defining K_i as $(1,1)$-block of $\left(A_i \tilde{\Phi}^{-1} A_i^\top \right)^{-1}$:

$$V(i) = \begin{bmatrix} \overline{x} \\ 0 \end{bmatrix}^\top \left(A_i \tilde{\Phi}^{-1} A_i^\top \right)^{-1} \begin{bmatrix} \overline{x} \\ 0 \end{bmatrix} = \overline{x}^\top K_i \overline{x}$$

Optimality of state feedback

Theorem: **For every state map $X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ and every $i \in [1, T_f]$ there exists a matrix $L_i \in \mathbb{R}^{\mathtt{w} \times \bullet}$ such that the optimal trajectory w^* satisfies**

$$w^*(i) = L_i \left(X(\sigma) w^* \right)(i)$$

Proof: **From**

$$\begin{bmatrix} w^*(i+L) \\ \vdots \\ w^*(T_f) \end{bmatrix} = \tilde{\Phi}^{-1} A_i^\top \left(A_i \tilde{\Phi}^{-1} A_i^\top \right)^{-1} \begin{bmatrix} \overline{x} \\ 0 \end{bmatrix}$$

and Bellman's optimality principle follows for all i

$$w^*(i) = \underbrace{\Phi^{-1} \begin{bmatrix} -R_L^\top & -R_{L-1}^\top & \cdots & -R_1^\top \end{bmatrix} K_i}_{=:L_i} \left(X(\sigma) w^* \right)(i)$$

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- ***LQ*-control solution**
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Thank you