Behavioral point-reachability

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Behaviors

$$\boldsymbol{\Sigma} = (\mathbb{T}, \mathbb{W}, \mathfrak{B}) \quad \mathbb{T} = \mathbb{Z}, \ \mathbb{W} = \mathbb{R}^{q}$$

$w \in \mathfrak{B}$ **iff** $R_N w(t+N) + \cdots + R_1 w(t+1) + R_0 w(t) = 0 \quad \forall t \in \mathbb{Z}$

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$$R(s) = R_N s^N + \cdots + R_1 s + R_0 \in \mathbb{R}^{\bullet \times q}[s] \subset \mathbb{R}^{\bullet \times q}\left[s, s^{-1}\right]$$

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$$\mathfrak{B} = \ker R(\sigma) := \left\{ w \in (\mathbb{R}^q)^{\mathbb{Z}} : R(\sigma) w = 0 \right\}$$

Kernel behavior

Example

$$\begin{cases} \boldsymbol{x}(\boldsymbol{t}+1) & = & \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t})+\boldsymbol{B}\boldsymbol{u}(\boldsymbol{t}) \\ \boldsymbol{y}(\boldsymbol{t}) & = & \boldsymbol{C}\boldsymbol{x}(\boldsymbol{t})+\boldsymbol{B}\boldsymbol{u}(\boldsymbol{t}) \end{cases} \tag{1}$$

Example

$$\begin{cases} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Bu(t) \end{cases} \qquad (2)$$

1) $\mathfrak{B}_{(x,u,y)} = \{w = (x, u, y) \mid (3) \text{ holds}\}$

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$$\begin{cases} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Bu(t) \end{cases} \tag{3}$$

1) $\mathfrak{B}_{(x,u,y)} = \{w = (x, u, y) \mid (3) \text{ holds}\}$

2) $\mathfrak{B}_{(u,y)} = \{w = (u, y) \mid \exists\ x \text{ s.t. } (3) \text{ holds}\}$

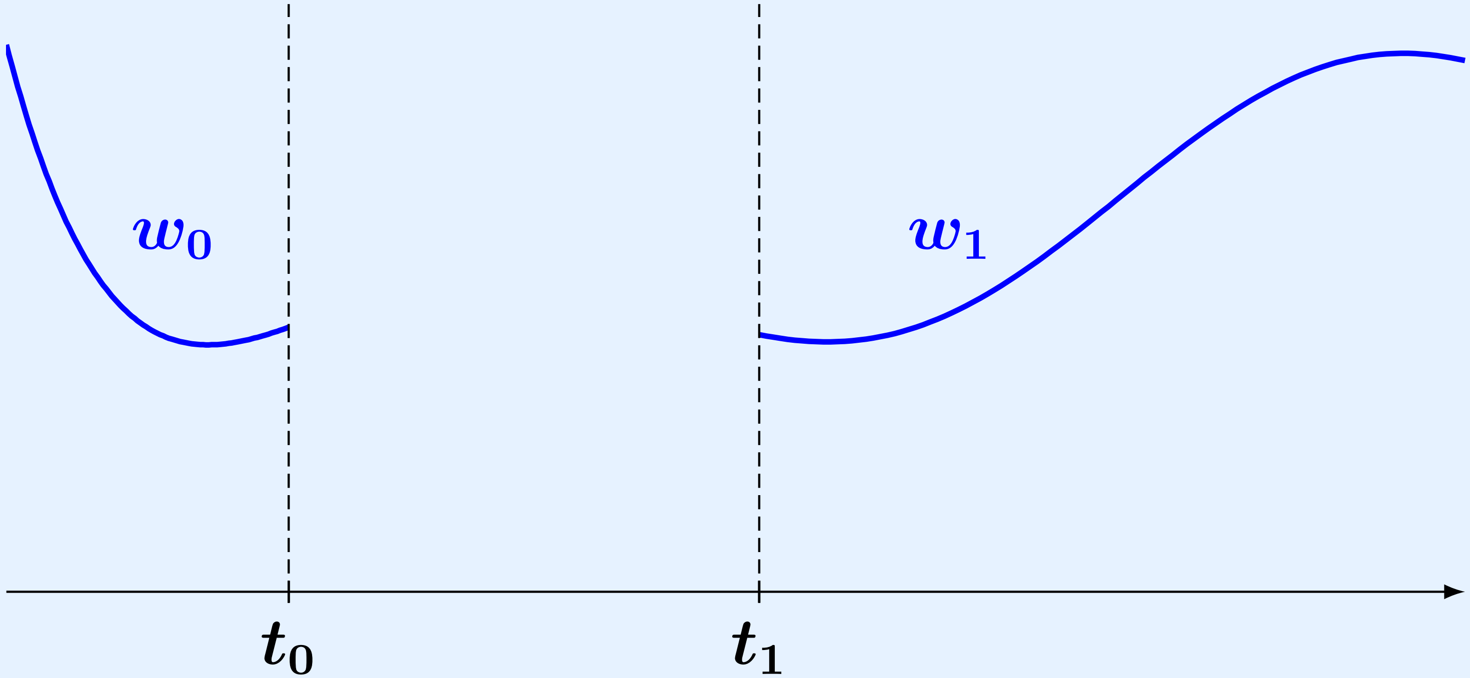
Controllability (Willems)

$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

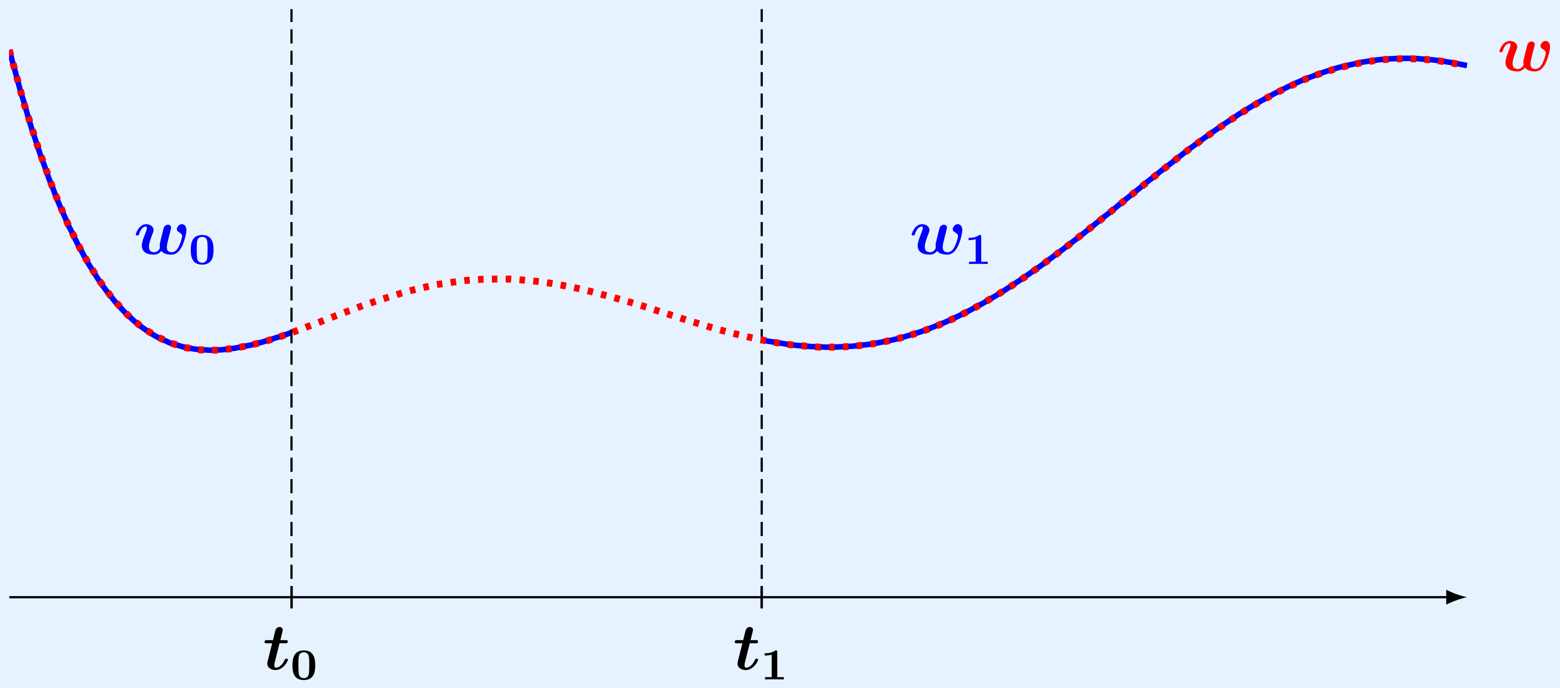
$\Sigma / \mathfrak{B}$ controllable

$\forall w_0, w_1 \in \mathfrak{B}$ **and** $\forall t_0 \in \mathbb{T}$, $\exists w \in \mathfrak{B}$ **and** $\exists t_1 \geq t_0$ **s.t.**

$$w(t) = \begin{cases} w_0(t), & t \leq t_0 \\ w_1(t), & t \geq t_1. \end{cases}$$



w_0
w_1
t_0
t_1



w
w_0
w_1
t_0
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Point-controllability (Willems)

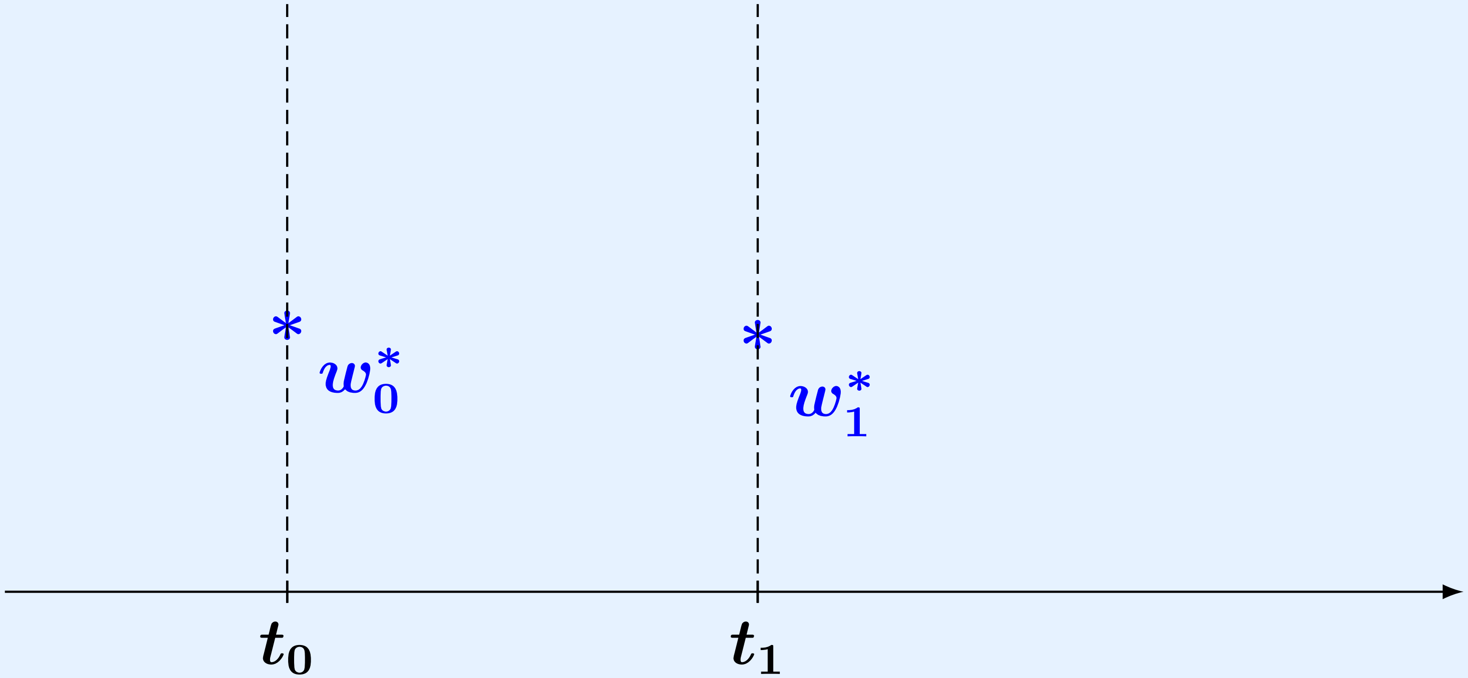
$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\Sigma/\mathfrak{B}$ **point-controllable**

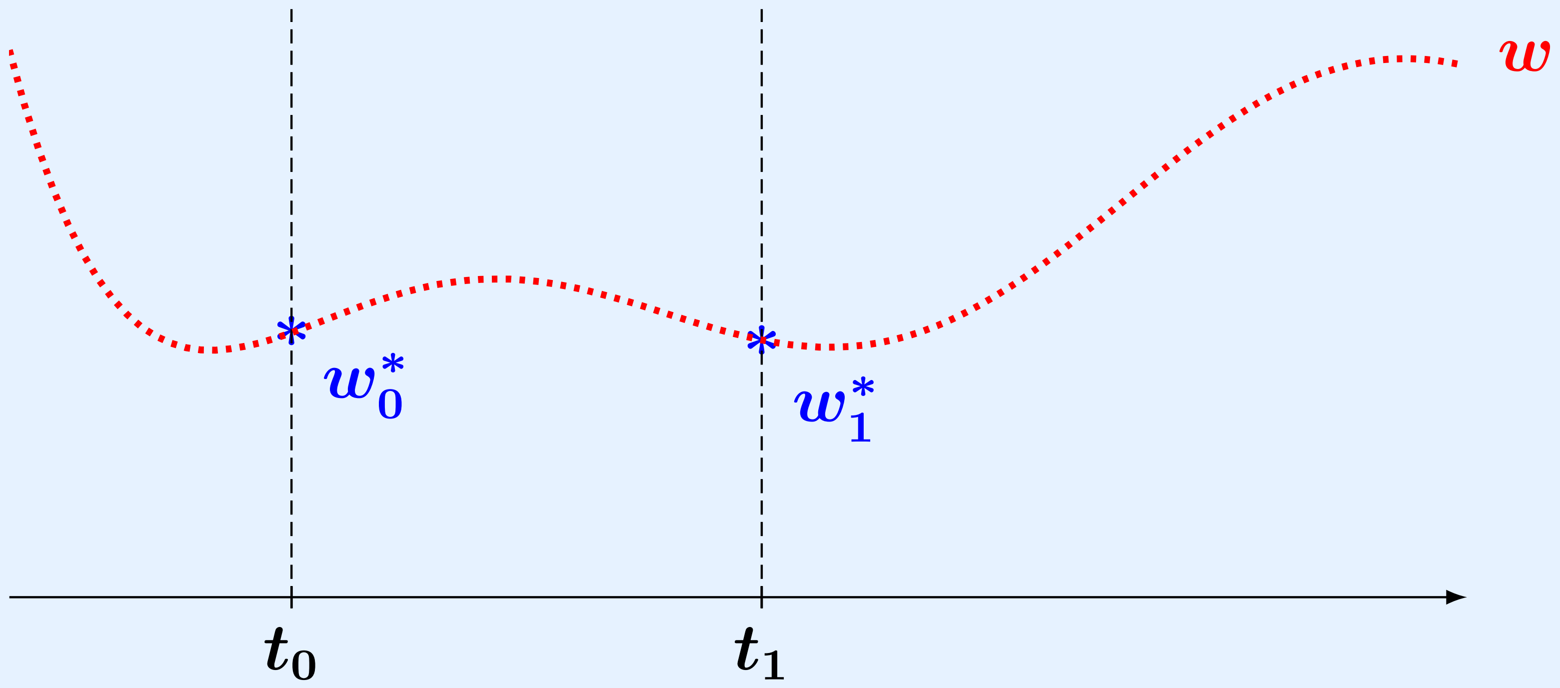
$\forall w_0^*, w_1^* \in \mathbb{W}$ **and** $\forall t_0 \in \mathbb{T}$, $\exists w \in \mathfrak{B}$ **and** $\exists t_1 \geq t_0$ **s.t.**

$$\begin{cases} w(t_0) = w_0^* \\ w(t_1) = w_1^*. \end{cases}$$

cf. state space systems



w_0^*
w_1^*
t_0
t_1


w
w_0^*
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Point-reachability

$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$w^* \in \mathbb{W}$ **reachable**

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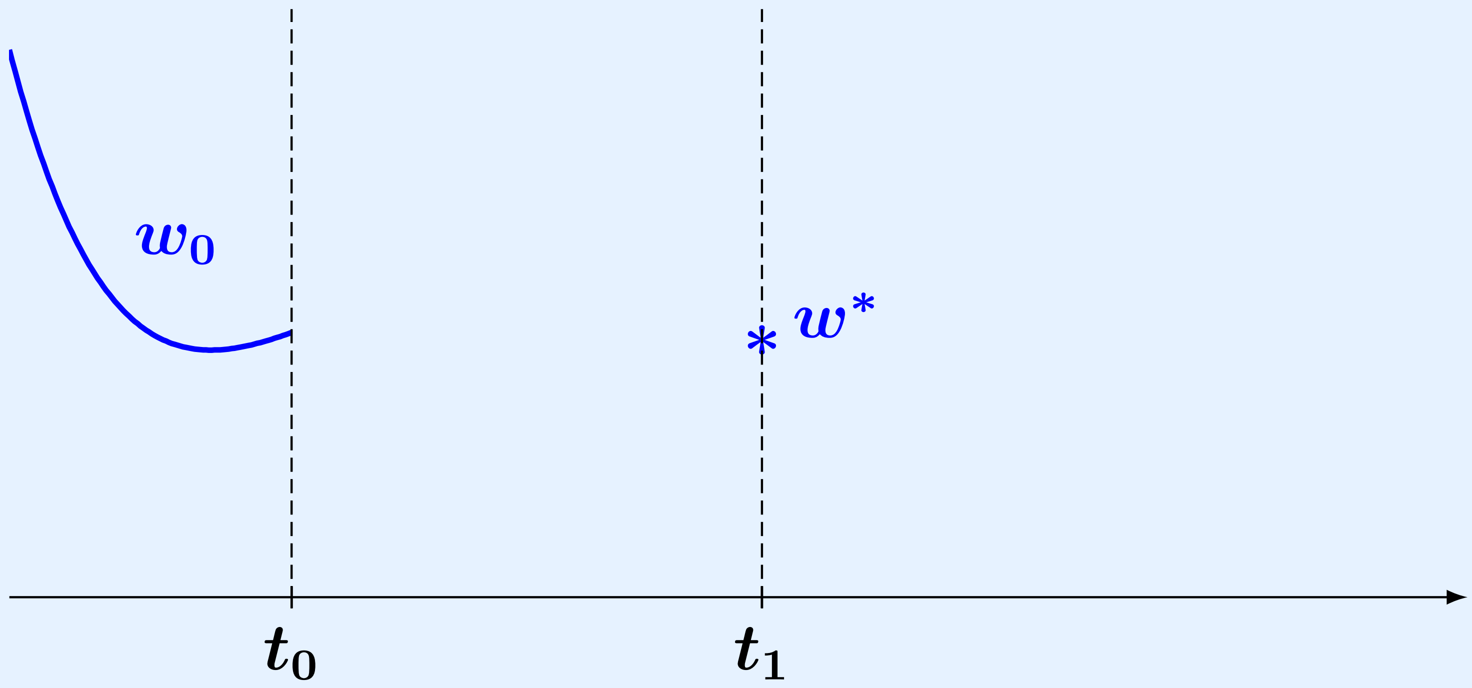
$\Sigma\ /\mathfrak{B}$ **point-reachable** - $\mathfrak{R}(\mathfrak{B}) = \mathbb{W}$


w_0
w^*
t_0
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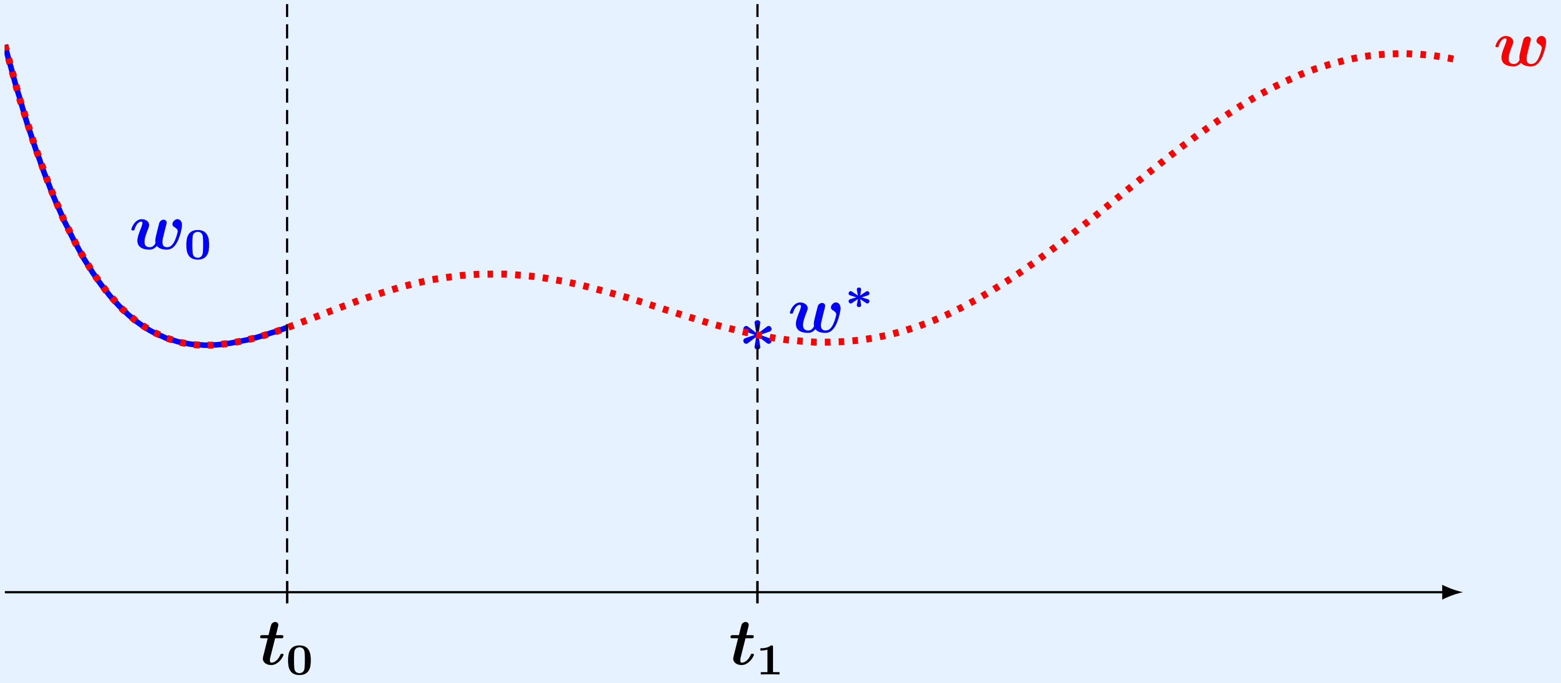

w
w_0
w^*
t_0
t_1

Point-reachability $\begin{matrix} \not\Rightarrow \\ \not\Leftarrow \end{matrix}$ **controllability**

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$\not\Rightarrow$: $\mathfrak{B} = \ker[\sigma - 1 \quad \sigma^2 - \sigma]$ **is point-reachable, but not controllable**

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$\not\Leftarrow$: $\mathfrak{B} = \{0\}$ **is controllable but not point-reachable**

Characterization of point-reachability

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Theorem 1 - Let $\mathfrak{B}$ be controllable, then

$$\mathfrak{B} \text{ is point-reachable } \Leftrightarrow \mathfrak{B} \text{ is trim (*)}$$

Characterization of point-reachability

First for controllable behaviors . . .

Theorem 1 - Let $\mathfrak{B}$ be controllable, then

$$\mathfrak{B} \text{ is point-reachable} \iff \mathfrak{B} \text{ is trim (*)}$$

(*) $\mathfrak{B}$ **trim** - $\forall w^* \in \mathbb{W} \;\; \exists w \in \mathfrak{B} \;\; \exists t^* \in \mathbb{Z} : w(t^*) = w^*$

no static laws

Example

$$\mathfrak{B} = \ker \begin{bmatrix} 1 & -(\sigma^2+\sigma+1) & (\sigma+1) \\ 0 & \sigma & -1 \end{bmatrix}$$

$$w = (w_1, w_2, w_3) \in \mathfrak{B} \Rightarrow w_1 = w_2$$

$\mathfrak{B}$ is **not trim**

Theorem (Willems)

$\mathfrak{B}$ controllable $\Leftrightarrow \exists\ M(s) = M_L s^L + \ldots + M_0$ s.t. $\mathfrak{B} = \mathrm{im}\ M(\sigma)$.

$w(t) = M_L a(t+L) + \ldots + M_0 a(t)$ for some a

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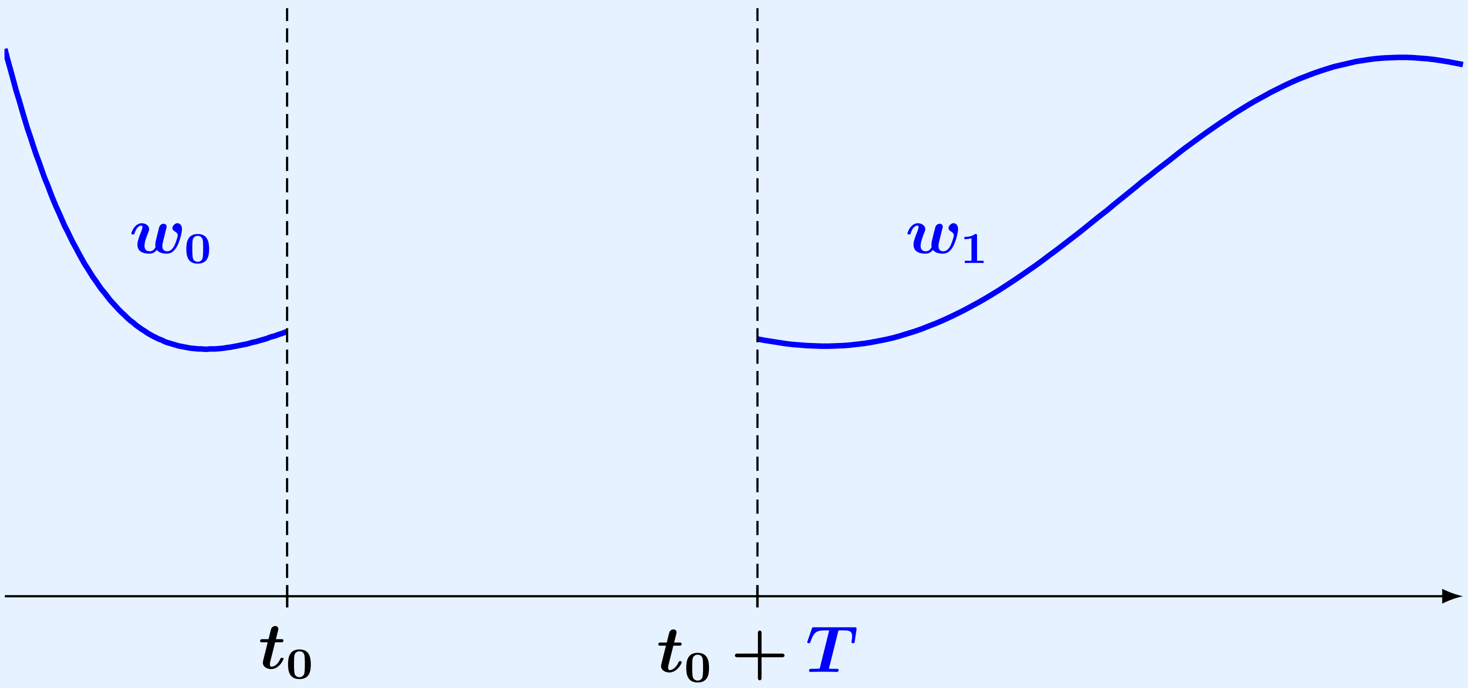

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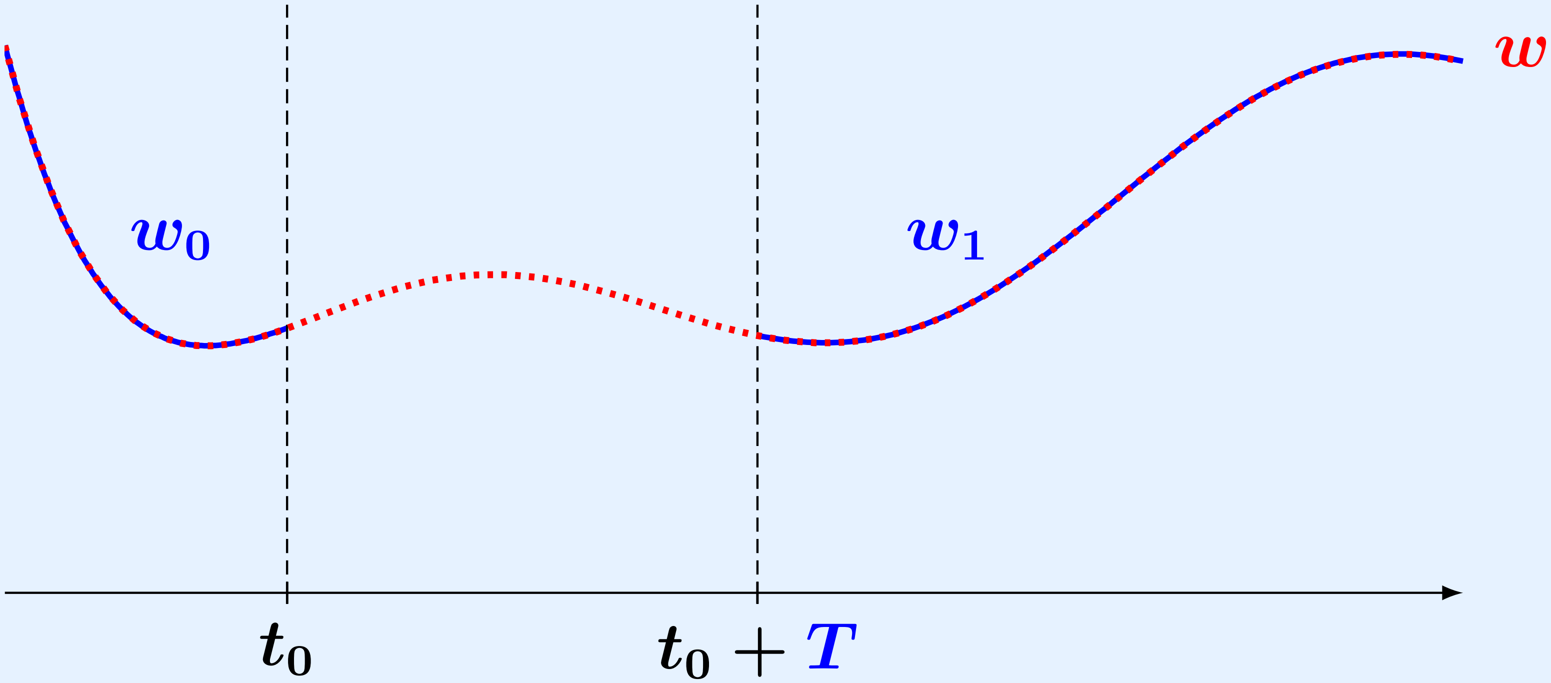

Example

$$\mathfrak{B} = \ker[\sigma^2 + 1 \quad 1 - \sigma]$$

$$w_1(t+2) + w_1(t) = w_2(t+1) - w_2(t)$$

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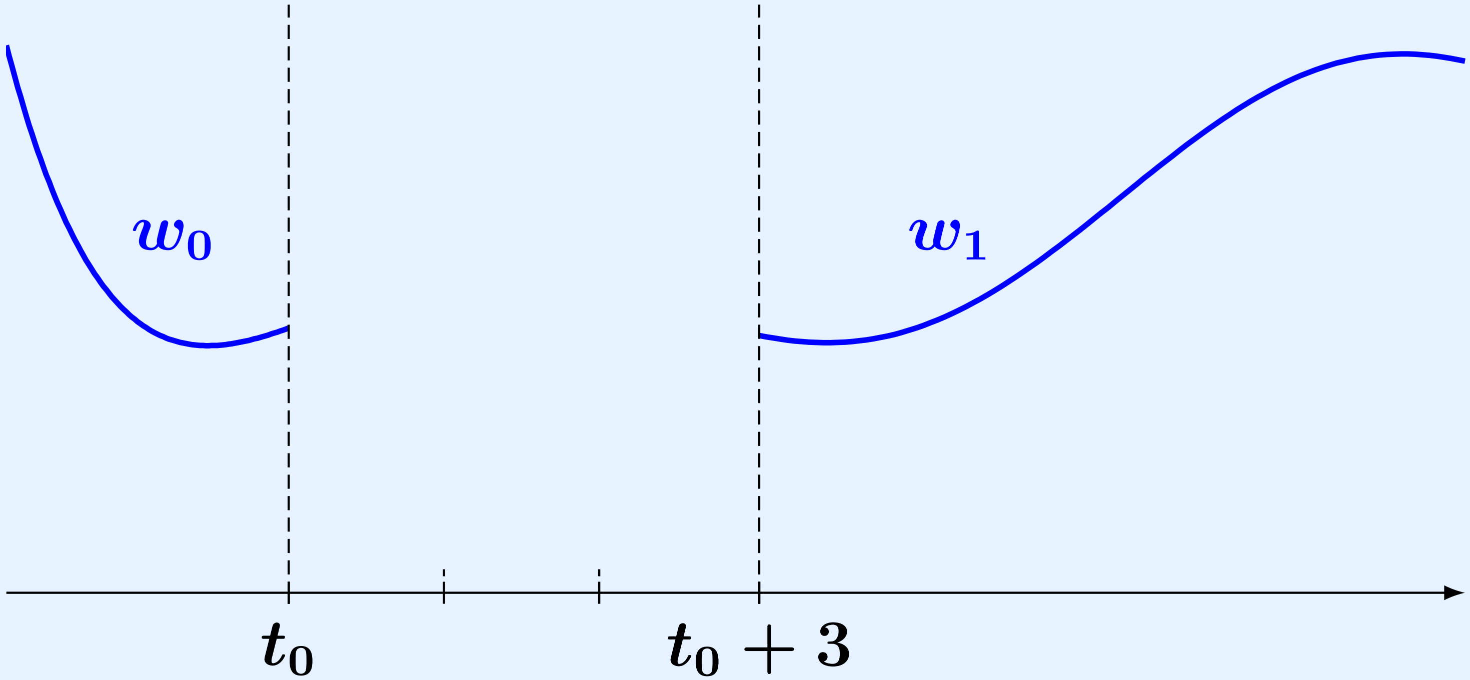
Control time - $T \geq 2 + 1 = 3$

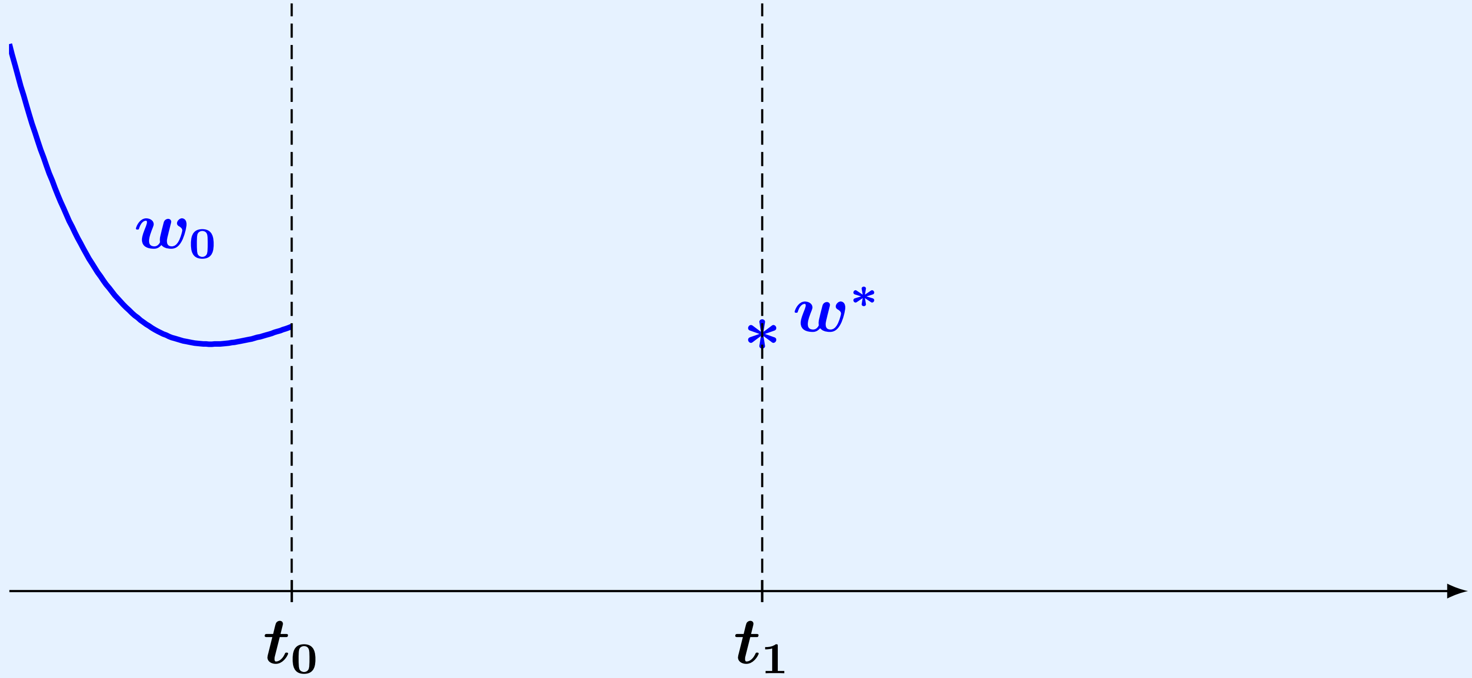
Example (cont.)



Proof of Theorem 1

$\Rightarrow$ **obvious**

$\Leftarrow$



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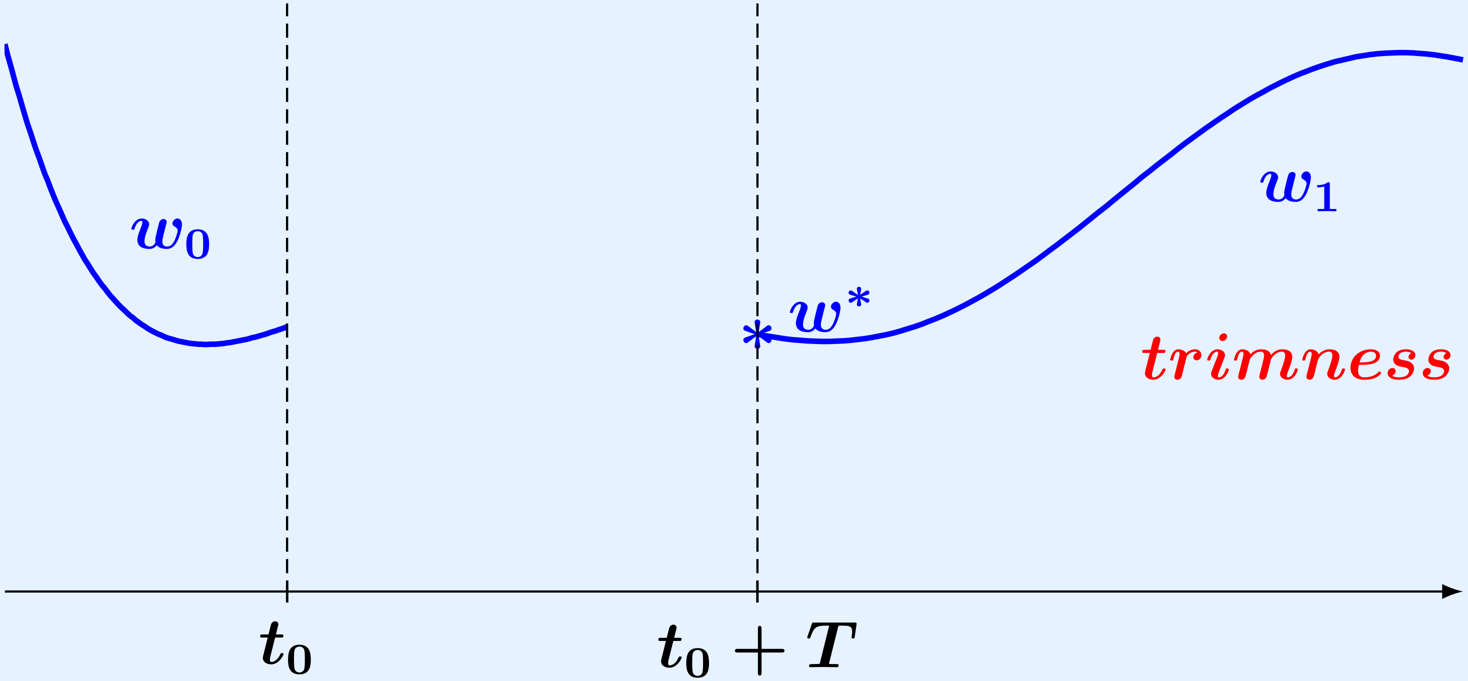

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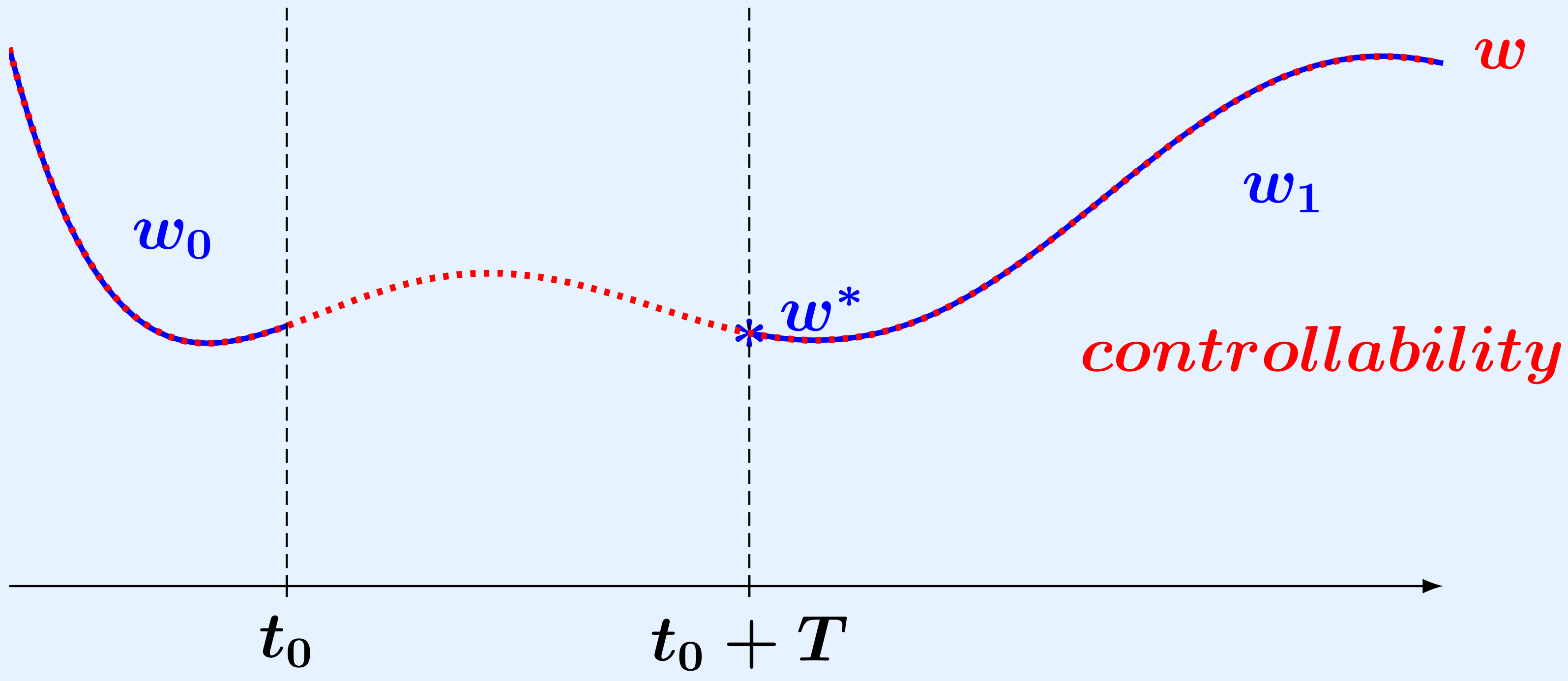

What about non-controllable behaviors?

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Controllable/autonomous decomposition

$\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^a$

$\mathfrak{B}^c$ - controllable part of $\mathfrak{B}$

$\mathfrak{B}^a$ - an autonomous* part of $\mathfrak{B}$

* autonomous = with no free variables

Theorem 2 - **The following statements are equivalent**

(i) $\mathfrak{B}$ **is point-reachable**

(ii) $\mathfrak{B}^c$ **is point-reachable**

(iii) $\mathfrak{B}^c$ **is trim**

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(ii) $\mathfrak{B}^c$ is point-reachable

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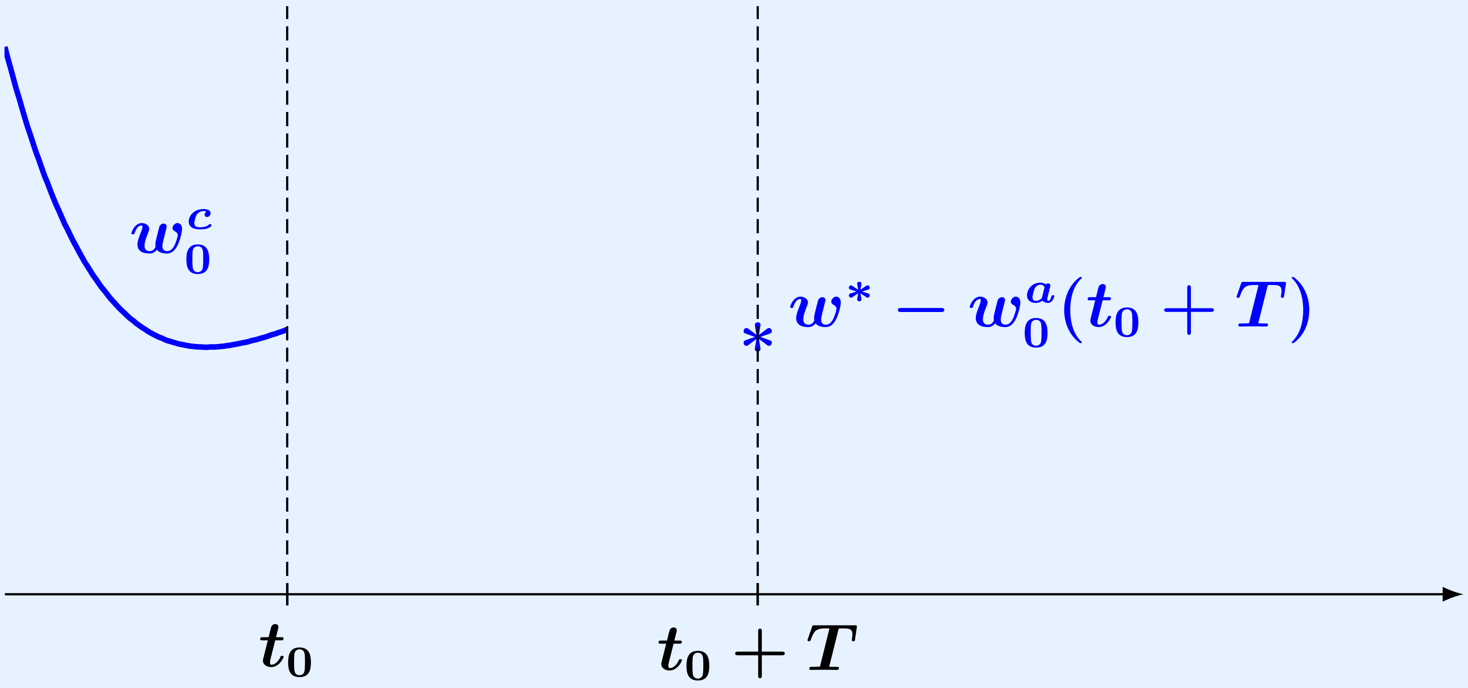
Proof:

(i) $\Rightarrow$ **(ii)** easy

(ii) $\Leftrightarrow$ **(iii)** Thm 1

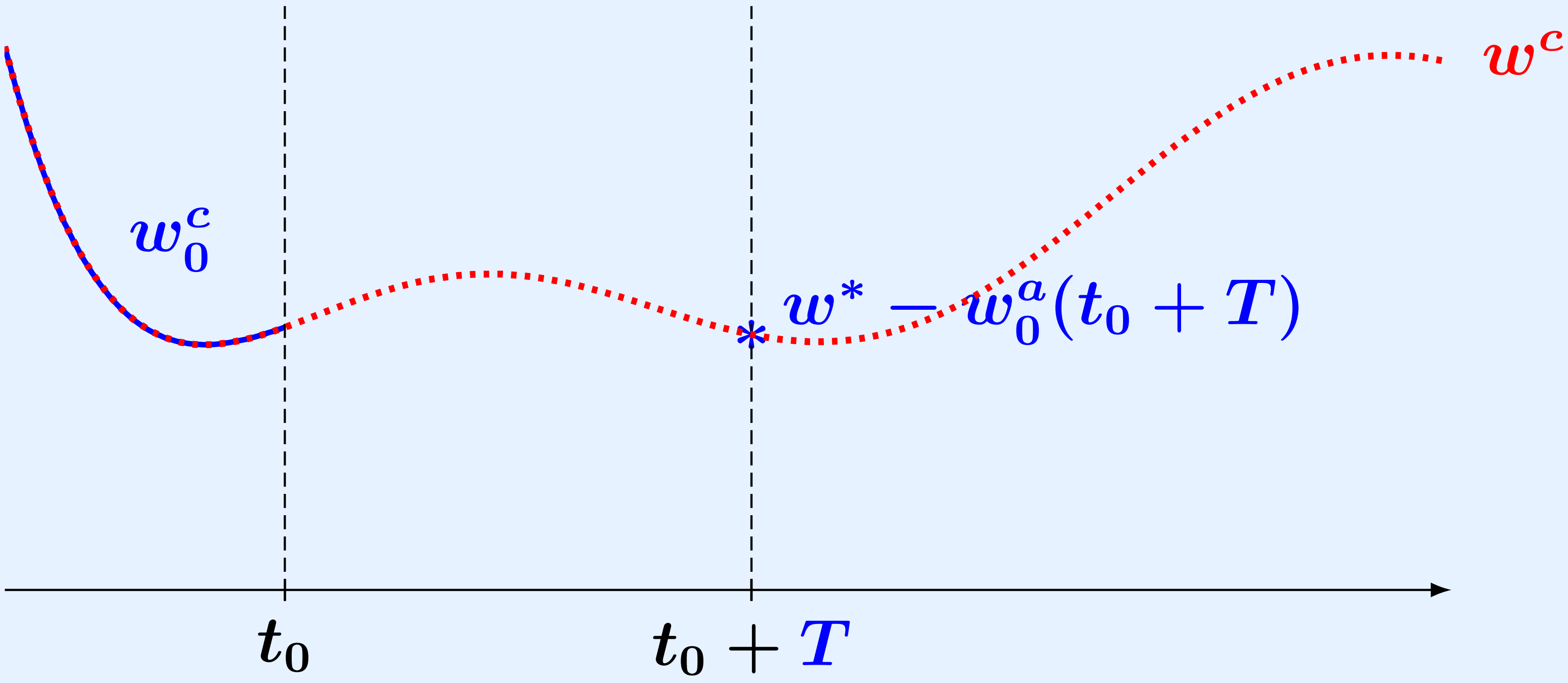
(ii) $\Rightarrow$ (i)

$$\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^a \qquad w_0 = w_0^c + w_0^a$$



(ii) $\Rightarrow$ (i)

$\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^a \qquad w_0 = w_0^c + w_0^a$



$w = w^c + w^a \;\longrightarrow$ **coincides with** w_0 **up to** t_0 **and** $w(t_0 + T) = w^*$

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  - **Determine a MRA of** $R(s)$ **-**

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- Form the matrix $K = [M_L | \ldots | M_0]$
- $\mathfrak{B}$ is point-reachable iff K has frr

Example

$$\mathfrak{B} = \ker(\sigma - 1)[\sigma + 1 \quad \sigma + 2]$$

$$\mathfrak{B}^c = \mathrm{im} \begin{bmatrix} -(\sigma + 2) \\ \sigma + 1 \end{bmatrix}$$

$$\mathrm{range}(\mathfrak{B}^c) = \mathrm{im} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \mathbb{R}^2$$

Thus $\mathfrak{B}^c$ is trim and $\mathfrak{B}$ is point-reachable

Proposition - $\mathfrak{B}$ point-reachable $\Rightarrow$ $\mathfrak{B}$ point-controllable

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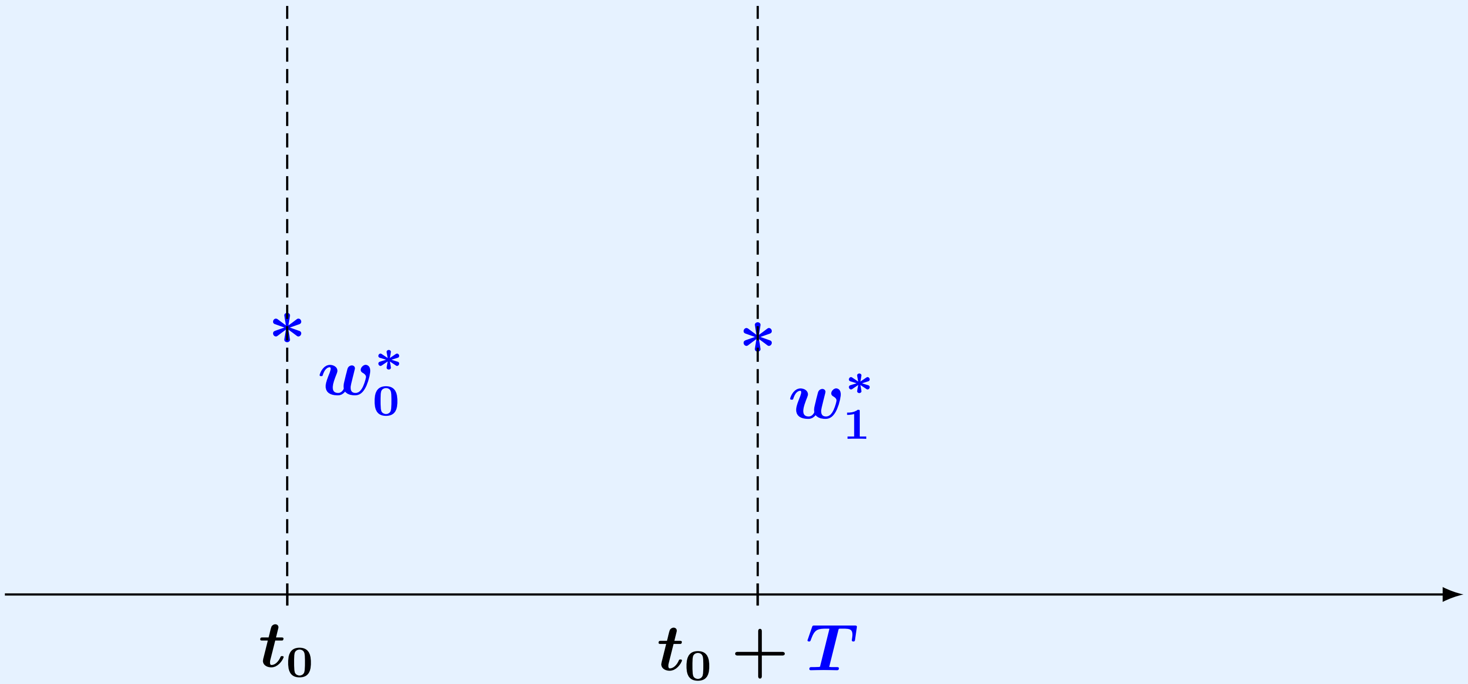
Proof - using the trimness of $\mathfrak{B}^c$



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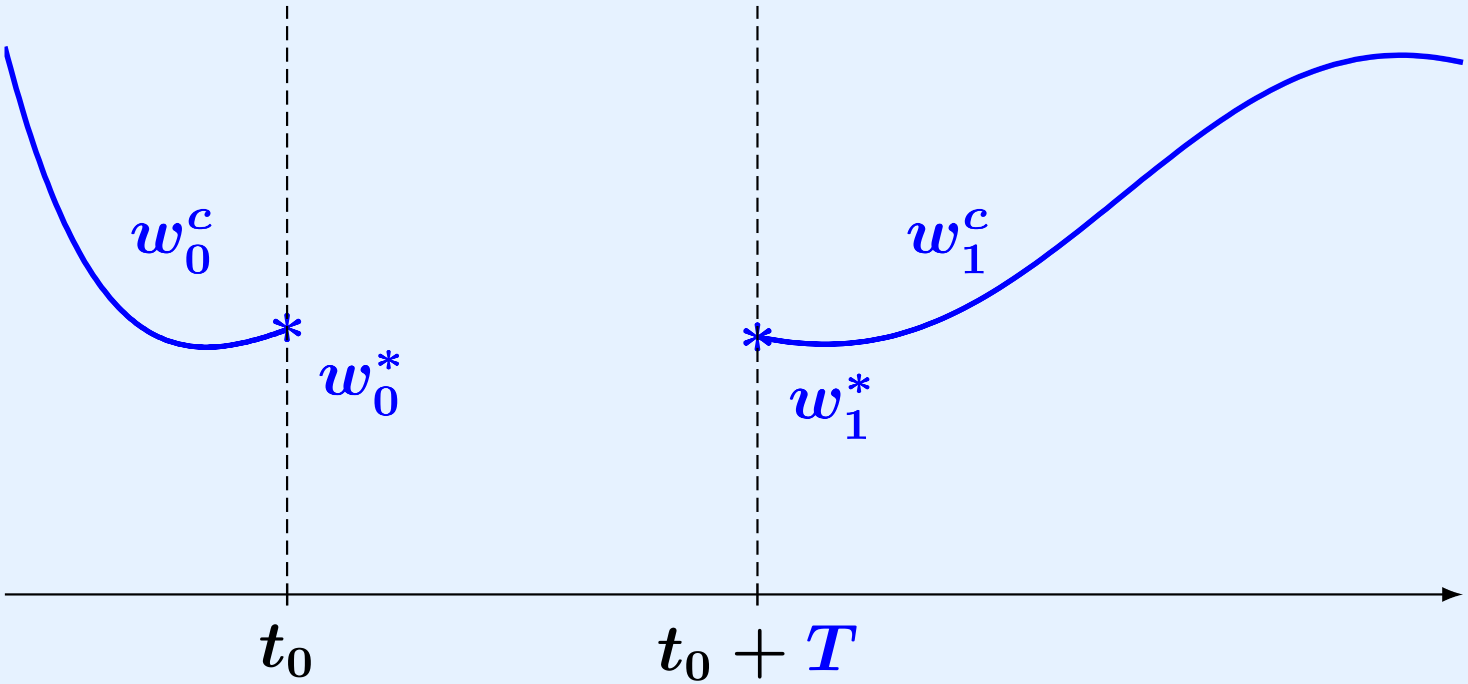

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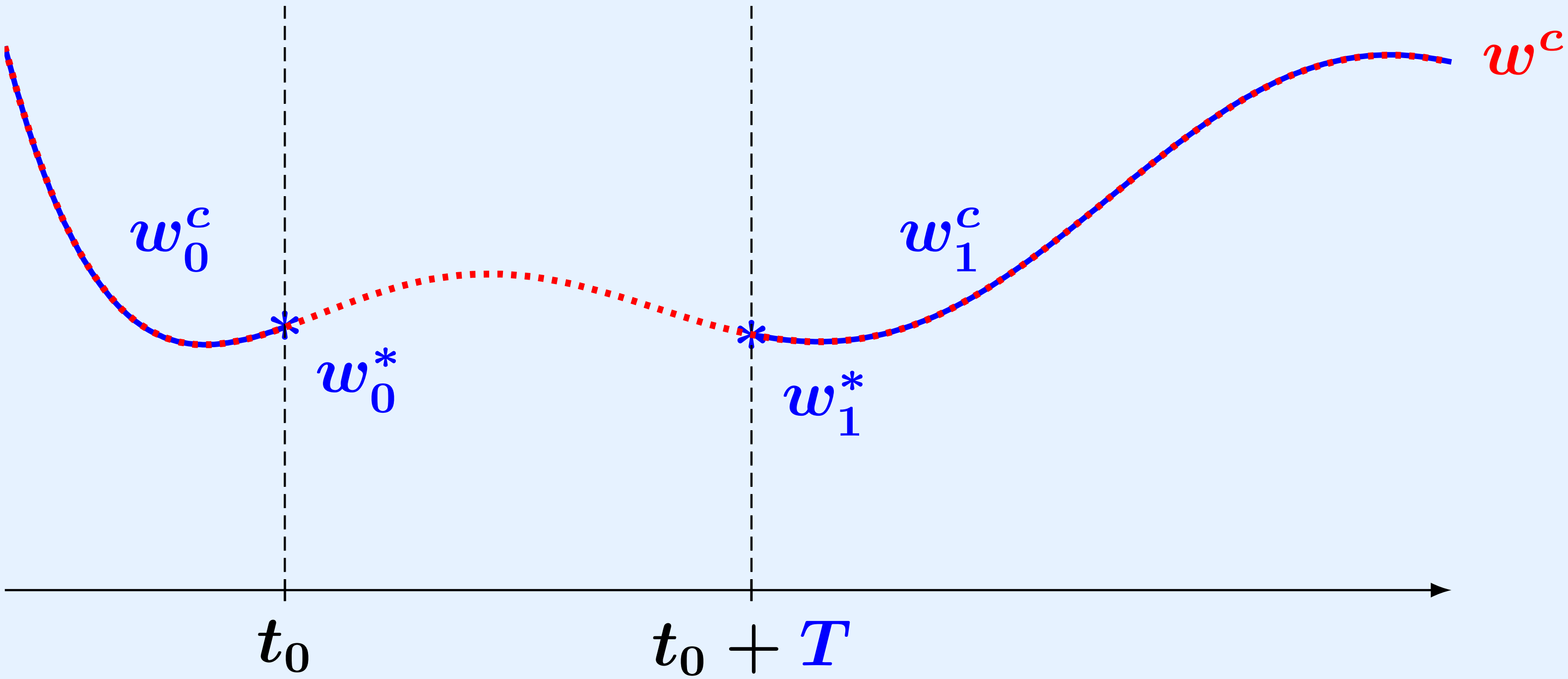

Example - $\mathfrak{B}$ point-reachable $\not\Leftarrow$ $\mathfrak{B}$ point-controllable

$$\mathfrak{B} = \ker(\sigma^3 - 1) \qquad w(t+3) = w(t)$$

$\mathfrak{B}$ is point-controllable but not point-reachable

For kernel behaviors

point-reachability

$\Updownarrow$

trimness of the controllable part

$\Downarrow$

point-controllability

behavioral controllability

For state behaviors

point-reachability

$\Updownarrow$

state space reachability

$\Updownarrow$

point-controllability

$\Updownarrow$

behavioral controllability and trimness

What happens when $\mathfrak{B}$ is not point-reachable?

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Characterization of $\mathfrak{R}(\mathfrak{B})$?

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Either $\mathfrak{R}(\mathfrak{B}) = \varnothing$ **or** $\mathfrak{R}(\mathfrak{B}) = \mathrm{range}(\mathfrak{B}^c)$

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Either $\mathfrak{R}(\mathfrak{B}) = \varnothing$ or $\mathfrak{R}(\mathfrak{B}) = \mathrm{range}(\mathfrak{B}^c)$

$$\mathfrak{R}(\mathfrak{B}) = \mathrm{range}(\mathfrak{B}^c) \Leftrightarrow \mathrm{range}(\mathfrak{B}^a) \subset \mathrm{range}(\mathfrak{B}^c)$$

Example $\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^a$

$$\mathfrak{B}^c = \mathrm{im} \begin{bmatrix} \sigma + 1 \\ 1 - \sigma \\ 2 \end{bmatrix}$$

$$\mathfrak{B}^a = \{w = (w_1, w_2, w_3) \mid (\sigma - 1)w_1 = 0,\ w_2 = w_1,\ w_3 = 2w_2\}$$

$$\mathrm{range}(\mathfrak{B}^c) = \{(w_1^*, w_2^*, w_3^*) \in \mathbb{R}^3 \mid w_3^* = w_1^* + w_2^*\}$$
$$\mathrm{range}(\mathfrak{B}^a) = \{(w_1^*, w_2^*, w_3^*) \in \mathbb{R}^3 \mid w_3^* = w_1^* + w_2^*,\ w_2 = w_1\}$$

Example (cont.)

$\mathbf{range}(\mathfrak{B}^a) \subset \mathbf{range}(\mathfrak{B}^c)$

Thus

$\mathfrak{R}(\mathfrak{B}) = \mathbf{range}(\mathfrak{B}^c) = \{(w_1^*, w_2^*, w_3^*) \in \mathbb{R}^3 \mid w_3^* = w_1^* + w_2^*\}$

Example $\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^a$

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Example (cont.)

$\mathrm{range}(\mathfrak{B}^a)$ is not contained in $\mathrm{range}(\mathfrak{B}^c)$

Thus $\mathfrak{R}(\mathfrak{B}) = \varnothing$