

# Generalized singular values

Ram Somaraju<sup>1</sup> and Jochen Trumpf<sup>2</sup>

We present a generalization of the usual Hilbert space concept of singular values for compact operators to the setting of arbitrary normed spaces.

The main idea behind the proposed generalization is to use finite dimensional exhaustion of the image  $W$  of the closed unit ball under the operator. For every given tolerance level  $\varepsilon > 0$  there is a minimal (finite) dimensional subspace of the image space approximating  $W$  with accuracy  $\varepsilon$  in norm. This minimal dimension is known as the “number of degrees of freedom at level  $\varepsilon$ ” in the theory of operator models for communication channels. The number of degrees of freedom is a non-increasing function of  $\varepsilon$  with a finite number of discontinuities in every given bounded  $\varepsilon$ -interval. We define the generalized singular values of the given operator to be the points of discontinuity of that function. This notion coincides with the usual concept of singular values in the special case of a Hilbert space setting.

We establish a number of basic results on generalized singular values, including some approximation results that lead to practical algorithms for their (approximate) computation. The latter require the existence of complete Schauder bases in the source and range spaces. We discuss an application to the question of fundamental capacity bounds on spatial communication channels as they appear in modern wireless multi-antenna systems. We will explain why Hilbert space models are not sufficiently general in this context and argue that compact operators between not necessarily complete normed spaces is the “correct” model class to use.

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<sup>1</sup>School of Information Technology and Electrical Engineering, UNSW@ADFA, Australian Defence Force Academy, [a.somaraju@gmail.com](mailto:a.somaraju@gmail.com).

<sup>2</sup>Corresponding author. Department of Information Engineering, The Australian National University, [Jochen.Trumpf@anu.edu.au](mailto:Jochen.Trumpf@anu.edu.au).