Consistency Analysis for Data Fusion: Determining When the Unknown Correlation Can be Ignored

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Abstract— **In this paper we examine the conditions in which data fusion can be performed by neglecting the unmodeled correlation between two information sources without compromising the consistency of the system. More specifically, we explore those situations in which one can disregard the correlation information and achieve a consistent estimate by simply adding the respective estimates' information matrices. This estimate will deliver considerably better performance than the widely employed Covariance Intersection (CI) algorithm in terms of estimation uncertainty.**

I. Introduction

In data fusion and estimation problems, multiple noise-corrupted variables are often combined (fused) together to obtain an improved estimate of some underlying state together with a measure of accuracy. A common assumption is that observations of a state are conditionally independent if the state is the *only* common underlying information. That is, once the state is known, observations become independent. However, this independence assumption cannot be guaranteed. For instance, two observations from a sensor platform mounted on a vehicle can be affected by the same process noise (e.g. be correlated through a common vibration). In addition, the observations can be estimates (rather than raw sensor measurements) that share prior information reported from a common information source. If the statistics of the correlations can be tracked down and identified, the full joint probability function [1] can be used to obtain the minimum mean squared error (MMSE) estimates of the target state. Otherwise, one of the many suboptimal approaches should be applied.

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Several methods have been developed to address the data fusion problem when exact knowledge of the correlation between information sources is not available. Methods based on Kalman filtering (KF) simply ignore the unmodeled correlations by assuming independence between the prior estimation error and the new information error. This presumption has been sufficient for a wide range of practical situations and has been successfully implemented in applications such as navigation [2], sensor fusion [3], map building [4] and target tracking [5]. Nevertheless, since the independence assumption is only an approximation to reality, it can potentially lead to serious problems. An example of such a case is the famous 'double counting' problem in distributed sensor networks which leads to information redundancy and over-confident estimates resulting from discarding the common information between two nodes; see [6]. In practice, a typical solution to avoid over-confident estimation relies on artificially inflating the covariance of the combined estimate. This method is ad-hoc and unreliable as the level of inflation cannot be precisely quantified and is largely application dependent.

Several works have tackled the inconsistency issue caused by ignoring the correlation by applying conservative fusion algorithms such as Covariance Intersection (CI). CI was first introduced in a seminal paper by Julier and Uhlmann [7] and has since been used in a wide spectrum of applications, particularly in the field of decentralised and distributed fusion [8]–[10]. The main benefit of using CI in data fusion applications is the ability of this algorithm to generate consistent estimates, regardless of the degree of correlation between the information sources. However, CI often results in highly conservative estimates, i.e. the estimated covariance can be much larger than the actual covariance.

In this paper we examine the conditions in which data fusion can be performed by neglecting the unmodeled correlation between two information sources without compromising the consistency of the system.

We explore those situations in which one can disregard the correlation information and achieve a consistent estimate by simply adding the respective estimates' information matrices. This estimate will deliver considerably better performance than the suboptimal CI.

This work is motivated by a practical project with the aim of developing a distributed fusion system to map a large-scale environment. The data fusion algorithm is distributed across multiple vehicles, each given the task of producing and updating a local map. The vehicles are equipped with a range of sensors and selectively communicate maps to and from a central station [8]. The local maps obtained from different vehicles are not independent; e.g. all vehicles share information obtained from the central station. Hence, an appropriate fusion strategy must be deployed to tackle the problem of correlated submaps.

The rest of this paper is arranged as follows: Section II provides some preliminaries on the data fusion problem under study. Section III outlines three classical fusion methods given correlated estimates. In Section IV conditions on consistent fusion, while ignoring the unknown correlation, will be derived for fusing two unbiased estimates. Simulations are provided in Section V and Conclusions are drawn in Section VI.

II. Preliminaries

We consider two (random variable) estimates $\mathbf{a} \sim \mathcal{N}(\mathbf{c}^*, \tilde{\mathbf{P}}_{aa})$ and $\mathbf{b} \sim \mathcal{N}(\mathbf{c}^*, \tilde{\mathbf{P}}_{bb})$ of some fixed parameter $\mathbf{c}^*$. The estimation error of $\mathbf{a}$ and $\mathbf{b}$ are defined by the random variables

$$\tilde{\mathbf{a}} = \mathbf{a} - \mathbf{c}^* \ , \ \tilde{\mathbf{b}} = \mathbf{b} - \mathbf{c}^* \tag{1}$$

where, in this case,

$$\mathrm{E}[\tilde{\mathbf{a}}] = 0 \ , \ \tilde{\mathbf{P}}_{aa} = \mathrm{E}[\tilde{\mathbf{a}}\tilde{\mathbf{a}}^\top] \tag{2}$$

$$\mathrm{E}[\tilde{\mathbf{b}}] = 0 \ , \ \tilde{\mathbf{P}}_{bb} = \mathrm{E}[\tilde{\mathbf{b}}\tilde{\mathbf{b}}^\top] \tag{3}$$

Although the true values $\tilde{\mathbf{P}}_{aa}$ and $\tilde{\mathbf{P}}_{bb}$ may not be known, consistent approximations $\mathbf{P}_{aa}$ and $\mathbf{P}_{bb}$ are assumed available where [1]

$$\mathbf{P}_{aa} \geq \tilde{\mathbf{P}}_{aa} \ , \ \mathbf{P}_{bb} \geq \tilde{\mathbf{P}}_{bb} \tag{4}$$

The cross-correlation matrix between the two estimates is denoted by $\tilde{\mathbf{P}}_{ab}$ and is defined by

$$\tilde{\mathbf{P}}_{ab} = \mathrm{E}[(\mathbf{a} - \mathbf{c}^*)(\mathbf{b} - \mathbf{c}^*)^\top] = \mathrm{E}[\tilde{\mathbf{a}}\tilde{\mathbf{b}}^\top] \tag{5}$$

This matrix may be known or unknown and may even be zero in some applications.

[1]This inequality is in the sense of matrix positive definiteness.

Let $\mathbf{c} \sim \mathcal{N}(\mathbf{c}^*, \mathbf{P}_{cc})$ denote a third estimate of $\mathbf{c}^*$ obtained via a linear combination of $\mathbf{a}$ and $\mathbf{b}$. That is

$$\mathbf{c} = \mathbf{K}_1\mathbf{a} + \mathbf{K}_2\mathbf{b} \tag{6}$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $\mathbf{K}_1, \mathbf{K}_2 \in \mathbb{R}^{n\times n}$. The error in this estimate is

$$\tilde{\mathbf{c}} = \mathbf{c} - \mathbf{c}^* \tag{7}$$

and obeys $\mathrm{E}[\tilde{\mathbf{c}}] = 0$ when $\mathbf{K}_1 + \mathbf{K}_2 = \mathbf{I}$.

The true covariance $\tilde{\mathbf{P}}_{cc} = \mathrm{E}[\tilde{\mathbf{c}}\tilde{\mathbf{c}}^\top]$ is calculated by

$$\tilde{\mathbf{P}}_{cc} = \mathbf{K}_1\tilde{\mathbf{P}}_{aa}\mathbf{K}_1^\top + \mathbf{K}_2\tilde{\mathbf{P}}_{bb}\mathbf{K}_2^\top + \mathbf{K}_1\tilde{\mathbf{P}}_{ab}\mathbf{K}_2^\top + \mathbf{K}_2\tilde{\mathbf{P}}_{ba}\mathbf{K}_1^\top \tag{8}$$

and calculation of this term requires $\tilde{\mathbf{P}}_{ab} = \tilde{\mathbf{P}}_{ba}^\top$ be known (when it is non-zero).

In this paper we are mainly interested in the construction of an estimate $\mathbf{P}_{cc}$ of $\tilde{\mathbf{P}}_{cc}$ when the cross-correlation $\tilde{\mathbf{P}}_{ab}$ is non-zero but unknown. We are further interested in certain properties of the resulting $\mathbf{P}_{cc}$. In particular, we are interested in the property of consistency

$$\mathbf{P}_{cc} \geq \tilde{\mathbf{P}}_{cc} \tag{9}$$

where $\tilde{\mathbf{P}}_{cc}$ is given by (8). In this case, (8) holds for any estimator defined by the linear combination (6) but the computation (8) requires knowledge of the cross-correlation $\tilde{\mathbf{P}}_{ab}$ or some estimation thereof.

In many cases, one is not interested in the class of estimators defined by arbitrary parameters $\mathbf{K}_1 + \mathbf{K}_2 = \mathbf{I}$ but rather in some optimal estimator. In this case, we note the following estimator defined by

$$(\mathbf{K}_1^*, \mathbf{K}_2^*) = \underset{(\mathbf{K}_1,\mathbf{K}_2)}{\mathrm{argmin}} \ \mathrm{tr}(\tilde{\mathbf{P}}_{cc}) \quad \text{s.t.} \quad \mathbf{K}_1 + \mathbf{K}_2 = \mathbf{I} \tag{10}$$

$$\tilde{\mathbf{P}}_{cc} = \begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{P}}_{aa} & \tilde{\mathbf{P}}_{ab} \\ \tilde{\mathbf{P}}_{ab}^T & \tilde{\mathbf{P}}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{K}_1^\top \\ \mathbf{K}_2^\top \end{bmatrix} \tag{11}$$

where the pair $\mathbf{K}_1$ and $\mathbf{K}_2$ are chosen to minimise the trace of $\tilde{\mathbf{P}}_{cc}$. Solving the above constrained optimisation problem for $\mathbf{K}_1$ and $\mathbf{K}_2$ yields an optimal value for $\tilde{\mathbf{P}}_{cc}$ in the form of

$$\tilde{\mathbf{P}}_{cc}^{*^{-1}} = \tilde{\mathbf{P}}_{aa}^{-1} + (\tilde{\mathbf{P}}_{aa}^{-1}\tilde{\mathbf{P}}_{ab} - \mathbf{I})(\tilde{\mathbf{P}}_{bb} - \tilde{\mathbf{P}}_{ab}^\top\tilde{\mathbf{P}}_{aa}^{-1}\tilde{\mathbf{P}}_{ab})^{-1} \cdot (\tilde{\mathbf{P}}_{ab}^\top\tilde{\mathbf{P}}_{aa}^{-1} - \mathbf{I}) \tag{12}$$

As noted, in this paper we are concerned, primarily with the construction of a consistent estimate $\mathbf{P}_{cc}$ of $\tilde{\mathbf{P}}_{cc}$ when the cross-correlation $\tilde{\mathbf{P}}_{ab}$ is non-zero but unknown. To this end we define consistency against the optimal value $\tilde{\mathbf{P}}_{cc}^*$ which in turn is defined as that $\tilde{\mathbf{P}}_{cc}$ with the minimum trace over all estimators of the form (6).

Definition 1. *Suppose $\tilde{\mathbf{P}}_{aa}$ and $\tilde{\mathbf{P}}_{bb}$ are given along with $\tilde{\mathbf{P}}_{ab} = \tilde{\mathbf{P}}_{ba}^\top$. Suppose $\tilde{\mathbf{P}}_{ab} = \tilde{\mathbf{P}}_{ba}^\top$ is non-zero. An estimate $\mathbf{P_{cc}}$ of $\tilde{\mathbf{P}}_{cc}$ is said to be consistent if*

$$\mathbf{P}_{cc} \geq \tilde{\mathbf{P}}_{cc}^* \tag{13}$$

where $\tilde{\mathbf{P}}_{cc}^*$ is an optimal value for $\tilde{\mathbf{P}}_{cc}$ given by (12).

This definition of consistency is particularly useful for the purposes of studying information fusion algorithms as it relates practical estimators (particularly their uncertainty estimate) with an ideal estimator that could be constructed if the cross-correlation between individual estimators were known (and it was known that individual estimates were not over-confident).

It is generally true that ignoring the correlation information $\tilde{\mathbf{P}}_{ab}$ when fusing $\mathbf{a}$ and $\mathbf{b}$ can lead to overly confident results; i.e. the resulting estimate of $\mathbf{P}_{cc}$ will be inconsistent as per Definition 1. Some algorithms, such as covariance intersection (CI), on the other hand are designed to generate consistent estimates when the cross-correlation is unknown. In many cases, the resulting estimators are considerably conservative. We explore those situations in which one can simply ignore the correlation information and achieve a consistent estimate by simply adding the respective estimates' information matrices. This estimate will deliver considerably better performance than the suboptimal covariance intersection. The specific details of the estimators in question will become clear as the paper progresses.

III. Three Classical Fusion Algorithms

In this section we outline three well-known estimation algorithms given the setup provided in the previous section. Each estimator assumes different information to be available for computation. We are mainly focused on the computation of the estimator's covariance in this paper as we will later be concerned with consistency.

A. Minimum Trace Fusion of Two Normally Distributed Estimators with a Known Degree of Correlation

We consider two estimates $\mathbf{a} \sim \mathcal{N}(\mathbf{c}^*, \tilde{\mathbf{P}}_{aa})$ and $\mathbf{b} \sim \mathcal{N}(\mathbf{c}^*, \tilde{\mathbf{P}}_{bb})$ of some fixed parameter $\mathbf{c}^*$. Suppose two consistent estimates of $\mathbf{a}$ and $\mathbf{b}$ with $\mathbf{P}_{aa} \geq \tilde{\mathbf{P}}_{aa}$ and $\mathbf{P}_{bb} \geq \tilde{\mathbf{P}}_{bb}$ are available and the cross-correlation $\tilde{\mathbf{P}}_{ab}$ is known. Replacing $\tilde{\mathbf{P}}_{aa}$ and $\tilde{\mathbf{P}}_{bb}$ in (12) by $\mathbf{P}_{aa}$ and $\mathbf{P}_{bb}$ respectively, automatically generates a consistent estimate $\mathbf{P}_{cc}^* \geq \tilde{\mathbf{P}}_{cc}^*$ if $\tilde{\mathbf{P}}_{ab} = \tilde{\mathbf{P}}_{ba}^\top$ is known. This is a consequence of Eq. (8). Therefore, when $\mathbf{P}_{aa}$ and $\mathbf{P}_{bb}$ are consistent and $\tilde{\mathbf{P}}_{ab}$ is known then the combined estimate

$$\mathbf{P}_{cc}^{*\,-1} = \mathbf{P}_{aa}^{-1} + (\mathbf{P}_{aa}^{-1}\tilde{\mathbf{P}}_{ab} - \mathbf{I})(\mathbf{P}_{bb} - \tilde{\mathbf{P}}_{ab}^\top \mathbf{P}_{aa}^{-1}\tilde{\mathbf{P}}_{ab})^{-1} \cdot (\tilde{\mathbf{P}}_{ab}^\top \mathbf{P}_{aa}^{-1} - \mathbf{I}) \tag{14}$$

is by definition consistent (as per Definition 1). As noted, the problem in practice is that $\tilde{\mathbf{P}}_{ab}$ is typically unknown.

B. Fusion of Two Normally Distributed Estimators with an Unknown Degree of Correlation: Covariance Intersection

In many practical applications the degree of correlation between different information sources is not available. A common solution in this case is to use the well-known covariance intersection (CI) algorithm. Suppose again we have two estimates $\mathbf{a} \sim \mathcal{N}(\mathbf{c}^*, \tilde{\mathbf{P}}_{aa})$ and $\mathbf{b} \sim \mathcal{N}(\mathbf{c}^*, \tilde{\mathbf{P}}_{bb})$ of some fixed parameter $\mathbf{c}^*$. Suppose consistent estimates $\mathbf{P}_{aa} \geq \tilde{\mathbf{P}}_{aa}$ and $\mathbf{P}_{bb} \geq \tilde{\mathbf{P}}_{bb}$ are available. The cross-correlation $\tilde{\mathbf{P}}_{ab}$ is unknown (cannot be used in the fusion algorithm) and may be non-zero. Then CI is defined by a convex combination

$$\mathbf{P}_{cc}^{\mathrm{CI}^{-1}} = \omega \mathbf{P}_{aa}^{-1} + (1-\omega)\mathbf{P}_{bb}^{-1} \tag{15}$$

$$\mathbf{P}_{cc}^{\mathrm{CI}^{-1}}\mathbf{c} = \omega \mathbf{P}_{aa}^{-1}\mathbf{a} + (1-\omega)\mathbf{P}_{bb}^{-1}\mathbf{b} \tag{16}$$

where $\mathbf{c} \sim \mathcal{N}(\mathbf{c}^*, \mathbf{P}_{cc})$ is an estimate of $\mathbf{c}^*$ and where $\omega \in (0, 1)$ is calculated according to some criteria; e.g. such as minimising the trace of the resulting covariance matrix $\mathbf{P}_{cc}^{\mathrm{CI}}$.

We note here simply that for all $\omega \in (0, 1)$, CI is guaranteed consistent as per Definition 1; i.e. $\mathbf{P}_{cc}^{\mathrm{CI}} \geq \tilde{\mathbf{P}}_{cc}^*$ and is often considerably conservative. We point to the literature [7] for further discussion of the CI algorithm and its consistency.

C. Fusion of Two Normally Distributed Estimators with an Unknown Degree of Correlation: Assuming Zero Correlation

Suppose again we have two estimates $\mathbf{a} \sim \mathcal{N}(\mathbf{c}^*, \mathbf{P}_{aa})$ and $\mathbf{b} \sim \mathcal{N}(\mathbf{c}^*, \mathbf{P}_{bb})$ of some fixed parameter $\mathbf{c}^*$ and each estimate is consistent; i.e. $\mathbf{P}_{aa} \geq \tilde{\mathbf{P}}_{aa}$ and $\mathbf{P}_{bb} \geq \tilde{\mathbf{P}}_{bb}$. The cross-correlation $\tilde{\mathbf{P}}_{ab}$ is unknown (cannot be used in the fusion algorithm) and may be non-zero. Let $\mathbf{c} \sim \mathcal{N}(\mathbf{c}^*, \mathbf{P}_{cc})$ denote an estimate of $\mathbf{c}^*$.

Now if $\mathbf{a}$ and $\mathbf{b}$ were in fact uncorrelated, then substituting $\tilde{\mathbf{P}}_{ab} = 0$ into (14) yields

$$\mathbf{P}_{cc}^{0\;-1} = \mathbf{P}_{aa}^{-1} + \mathbf{P}_{bb}^{-1} \tag{17}$$

which can be computed and is subsequently (by definition) consistent as per Definition 1. We also have

$$\mathbf{P}_{cc}^{0\;-1}\mathbf{c} \;\; = \;\; \mathbf{P}_{aa}^{-1}\mathbf{a} + \mathbf{P}_{bb}^{-1}\mathbf{b} \tag{18}$$

for completeness. This solution is optimal (in the sense of a minimum trace) when $\tilde{\mathbf{P}}_{ab}$ is indeed zero.

The main question motivating the subsequent work in this paper is summarised in the following.

Question 1. *If one computes $\mathbf{P}_{cc}^{0\;-1} = \mathbf{P}_{aa}^{-1} + \mathbf{P}_{bb}^{-1}$ when $\tilde{\mathbf{P}}_{ab}$ is non-zero, is $\mathbf{P}_{cc}^{0}$ consistent as per Definition 1?*

It is easily observed that $\mathbf{P}_{cc}^{0} \leq \mathbf{P}_{cc}^{\mathrm{CI}}$. Thus, if $\mathbf{P}_{cc}^{0} \geq \tilde{\mathbf{P}}_{cc}^{*}$, i.e. if $\mathbf{P}_{cc}^{0}$ is consistent as per Definition 1, then it follows that estimation via $\mathbf{P}_{cc}^{0}$ is typically more desirable than estimation via $\mathbf{P}_{cc}^{\mathrm{CI}}$. It will turn out that the inequality $\mathbf{P}_{cc}^{0} \geq \tilde{\mathbf{P}}_{cc}^{*}$ holds for only some values of $\tilde{\mathbf{P}}_{ab}$. In those cases, it so happens that one may simply ignore (set to zero) the cross-correlation and perform optimal (minimum trace) fusion. The result will be sub-optimal (as expected) but better (in terms of the trace) than covariance intersection. The result, as per the definition of consistency, will be conservative (non-optimistic) as desired.

IV. Condition on consistent estimation under unknown correlation

It is well known that the CI algorithm guarantees the combined estimate to be consistent as per Definition 1. However, the consistency of $\mathbf{P}_{cc}^{0\;-1} = \mathbf{P}_{aa}^{-1} + \mathbf{P}_{bb}^{-1}$, i.e. simply ignoring the correlation, when $\tilde{\mathbf{P}}_{ab}$ is non-zero has yet to be established. As per Definition 1 consistency requires

$$\mathbf{P}_{cc}^{0} \geq \tilde{\mathbf{P}}_{cc}^{*} \tag{19}$$

where $\tilde{\mathbf{P}}_{cc}^{*}$ is computed by (12).

Now given consistent estimates $\mathbf{P}_{aa}$ and $\mathbf{P}_{bb}$ and a known cross-correlation $\tilde{\mathbf{P}}_{ab}$, a consistent representation of the combined estimate $\mathbf{P}_{cc}^{*}$ can be computed using Eq. (14). As explained in Subsection III-A, the resulting estimate automatically generates a consistent estimate, i.e.

$$\mathbf{P}_{cc}^{*} \geq \tilde{\mathbf{P}}_{cc}^{*} \tag{20}$$

As a consequence of (19) and (20), if the inequality

$$\mathbf{P}_{cc}^{0} \geq \mathbf{P}_{cc}^{*} \tag{21}$$

holds, the consistency of $\mathbf{P}_{cc}^{0}$ can be guaranteed as per Definition 1.

A. Consistency Analysis in One-Dimension

Suppose we have two estimates $a \sim \mathcal{N}(c^*, \tilde{P}_{aa})$ and $b \sim \mathcal{N}(c^*, \tilde{P}_{bb})$ of some fixed parameter $c^* \in \mathbb{R}$. Consistent estimates of a and b with $P_{aa} \geq \tilde{P}_{aa}$ and $P_{bb} \geq \tilde{P}_{bb}$ are available. The cross-correlation $\tilde{P}_{ab}$ is unknown (cannot be used in the fusion algorithm) and is non-zero. The following is the main result of this subsection.

Theorem 1. *Suppose one computes*

$$P_{cc}^{0\;-1} \;\; = \;\; P_{aa}^{-1} + P_{bb}^{-1} \tag{22}$$

$$P_{cc}^{0\;-1}c \;\; = \;\; P_{aa}^{-1}a + P_{bb}^{-1}b \tag{23}$$

Then,

$$P_{cc}^{0} \geq P_{cc}^{*} \tag{24}$$

if and only if

$$-\sqrt{P_{aa}P_{bb}} \leq \tilde{P}_{ab} \leq 0, \quad \text{or} \tag{25}$$

$$\left(\frac{P_{aa}^{-1} + P_{bb}^{-1}}{2}\right)^{-1} \leq \tilde{P}_{ab} \leq \sqrt{P_{aa}P_{bb}} \tag{26}$$

where P_{cc}^{*} is computed via (14) using the consistent $P_{aa} \geq \tilde{P}_{aa}$ and $P_{bb} \geq \tilde{P}_{bb}$ and the true $\tilde{P}_{ab}$.

That is in particular, P_{cc}^{0} is consistent as per Definition 1 when $\tilde{P}_{ab}$ obeys one of the theorem's stated inequalities.

Proof: The inequality (24) can be written as

$$P_{cc}^{0\;-1} \leq P_{cc}^{*\;-1}$$

$$P_{aa}^{-1} + P_{bb}^{-1} \leq P_{aa}^{-1} + (P_{aa}^{-1}\tilde{P}_{ab} - 1)(P_{bb} - \tilde{P}_{ab}^{\top}P_{aa}^{-1}\tilde{P}_{ab})^{-1}(\tilde{P}_{ab}^{\top}P_{aa}^{-1} - 1)$$

$$P_{aa}^{-1} + P_{bb}^{-1} \leq \frac{P_{aa} + P_{bb} - 2\tilde{P}_{ab}}{P_{aa}P_{bb} - \tilde{P}_{ab}^{2}}$$

$$(P_{aa} + P_{bb})(P_{aa}P_{bb} - \tilde{P}_{ab}^{2}) \leq P_{aa}P_{bb}(P_{aa} + P_{bb} - 2\tilde{P}_{ab})$$

Rearranging gives

$$\tilde{P}_{ab}\left[(P_{aa} + P_{bb})\tilde{P}_{ab} - 2P_{aa}P_{bb}\right] \geq 0$$

and thus

$$\tilde{P}_{ab} \leq 0, \text{ or } \tilde{P}_{ab} \geq \frac{2P_{aa}P_{bb}}{(P_{aa} + P_{bb})}$$

However, the joint covariance matrix

$$\mathbf{P} = \begin{bmatrix} P_{aa} & \tilde{P}_{ab} \\ \tilde{P}_{ab}^{\top} & P_{bb} \end{bmatrix}$$

must be positive definite which yields the upper and lower bounds on $\tilde{P}_{ab}$ and gives

$$-\sqrt{P_{aa}P_{bb}} \leq \tilde{P}_{ab} \leq 0, \quad \text{or}$$
$$\left(\frac{P_{aa}^{-1}+P_{bb}^{-1}}{2}\right)^{-1} \leq \tilde{P}_{ab} \leq \sqrt{P_{aa}P_{bb}}$$

This completes the proof.

This theorem suggests that if the ignored correlation $\tilde{P}_{ab}$ obeys the inequalities stated in the theorem then the solution provided by P_{cc}^{0} will still deliver a consistent estimate. An important point here is that P_{cc}^{0} is always smaller than P_{cc}^{CI} regardless of the correlation and thus offers a higher quality estimate. We state an equivalent result in a different way via the following corollary.

Corollary 1. *Consider the same one-dimensional problem setup as applied in the preceding theorem. For all consistent P_{aa} and P_{bb} there exists a choice of $\tilde{P}_{ab} \neq 0$ such that $P_{cc}^{0} > P_{cc}^{*}$ holds with strict inequality. Similarly, for all P_{aa} and P_{bb} there exists a different choice of $\tilde{P}_{ab} \neq 0$ such that $P_{cc}^{0} < P_{cc}^{*}$ holds with strict inequality.*

B. Consistency Analysis in Higher Dimensions

Consider two n-dimensional estimates ($n \in \mathbb{N}$) $\mathbf{a} \sim \mathcal{N}(\mathbf{c}^{*}, \tilde{\mathbf{P}}_{aa})$ and $\mathbf{b} \sim \mathcal{N}(\mathbf{c}^{*}, \tilde{\mathbf{P}}_{bb})$ of some fixed parameter $\mathbf{c}^{*} \in \mathbb{R}$. We consider a special case where consistent estimates $\mathbf{P}_{aa} \geq \tilde{\mathbf{P}}_{aa}$ and $\mathbf{P}_{bb} \geq \tilde{\mathbf{P}}_{bb}$ are available and are defined in the form of:

$$\mathbf{P}_{aa} = \gamma_a \cdot \mathbf{I}_n \tag{27}$$
$$\mathbf{P}_{bb} = \gamma_b \cdot \mathbf{I}_n \tag{28}$$

where γ_a and γ_b are scalars and $\mathbf{I}_n$ denotes the $(n \times n)$ identity matrix. The cross-correlation $\tilde{\mathbf{P}}_{ab}$ is unknown but assumed to be in the form of

$$\tilde{\mathbf{P}}_{ab} = \rho \cdot \mathbf{I}_n \tag{29}$$

where ρ is the scalar correlation coefficient. The following theorem summarises the main result of this subsection.

Theorem 2. *Suppose one computes*

$$\mathbf{P}_{cc}^{0\ -1} = \mathbf{P}_{aa}^{-1} + \mathbf{P}_{bb}^{-1} \tag{30}$$
$$\mathbf{P}_{cc}^{0\ -1}\mathbf{c} = \mathbf{P}_{aa}^{-1}\mathbf{a} + \mathbf{P}_{bb}^{-1}\mathbf{b} \tag{31}$$

Then,

$$\mathbf{P}_{cc}^{0} \geq \mathbf{P}_{cc}^{*} \tag{32}$$

if and only if

$$-\sqrt{\gamma_a\gamma_b} \leq \rho \leq 0, \quad \text{or} \tag{33}$$
$$\left(\frac{\gamma_a{}^{-1}+\gamma_b{}^{-1}}{2}\right)^{-1} \leq \rho \leq \sqrt{\gamma_a\gamma_b} \tag{34}$$

where $\mathbf{P}_{cc}^{*}$ is computed via (14) using the consistent $\mathbf{P}_{aa} \geq \tilde{\mathbf{P}}_{aa}$ and $\mathbf{P}_{bb} \geq \tilde{\mathbf{P}}_{bb}$ and the true $\tilde{\mathbf{P}}_{ab} = \rho \cdot \mathbf{I}_n$.

That is, $\mathbf{P}_{cc}^{0}$ is consistent as per Definition 1 when ρ obeys one of the inequalities in Equations (33) and (34). The proof for theorem 2 is fundamentally similar to the proof provided for the one-dimensional case in theorem 1, thus not provided here to avoid repetition.

This theorem suggests that if ρ in (29) obeys the inequalities stated in the theorem then the solution provided by $\mathbf{P}_{cc}^{0}$ will still deliver a consistent estimate. An important point here is that $\mathbf{P}_{cc}^{0}$ is always smaller than $\mathbf{P}_{cc}^{\mathrm{CI}}$ of (15) regardless of the correlation and thus offers a higher quality estimate. Similar to the one-dimensional case, we state an equivalent result via the following corollary.

Corollary 2. *Consider the same n-dimensional problem setup as applied in the preceding theorem. For all consistent $\mathbf{P}_{aa}$ and $\mathbf{P}_{bb}$ there exists a choice of $\tilde{\mathbf{P}}_{ab} \neq 0$ such that $\mathbf{P}_{cc}^{0} > \mathbf{P}_{cc}^{*}$ holds with strict inequality. Similarly, for all $\mathbf{P}_{aa}$ and $\mathbf{P}_{bb}$ there exists a different choice of $\tilde{\mathbf{P}}_{ab} \neq 0$ such that $\mathbf{P}_{cc}^{0} < \mathbf{P}_{cc}^{*}$ holds with strict inequality.*

V. Simulation

We now provide two simulations to exemplify the theorems stated in Section IV. The first simulation considers the fusion of two unbiased one-dimensional estimates $a \sim \mathcal{N}(0, \tilde{P}_{aa})$ and $b \sim \mathcal{N}(0, \tilde{P}_{bb})$ into estimate c. The covariances of the input estimates are given by $P_{aa} = 1$ and $P_{bb} = 0.3$. Fig. 1 compares the covariance of the combined estimate c as a function of the cross-correlation $\tilde{P}_{ab}$ for the three classical methods outlined in Section III. If the ignored correlation $\tilde{P}_{ab}$ satisfies the inequalities (25) and (26), the covariance of the obtained estimate P_{cc}^{0} is greater than the covariance of the optimal estimate provided by (14), thus guaranteeing a consistent estimate. The covariance of the solution obtained by using CI is always greater than both P_{cc}^{*} and P_{cc}^{0}.

Fig. 2 shows the fusion of two unbiased two-dimensional estimates $\mathbf{a}$ and $\mathbf{b}$ represented by $\mathbf{P}_{aa}$ and $\mathbf{P}_{bb}$ where

$$\mathbf{P}_{aa} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{P}_{bb} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

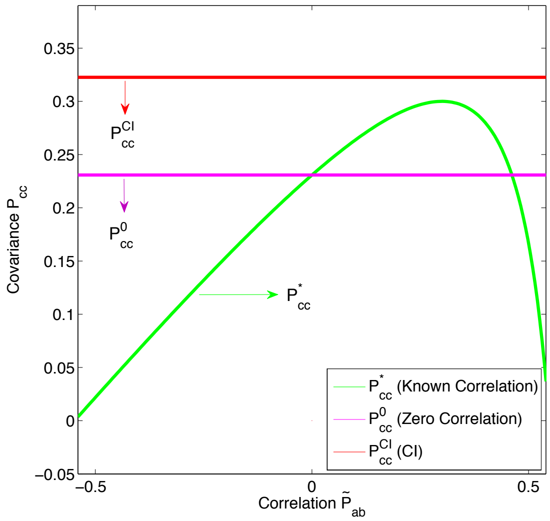

Fig. 1. Comparison of the covariance of the combined estimate as a function of the true cross-correlation $\tilde{P}_{ab}$. For those values of $\tilde{P}_{ab}$ where P_{cc}^0 is larger than the optimal value P_{cc}^*, consistent estimates can be achieved when the correlation is ignored. The intersection points of P_{cc}^0 and P_{cc}^* can be found by looking at the boundaries in (25) and (26). As expected, the conservative CI estimate P_{cc}^{CI} is always larger than both P_{cc}^* and P_{cc}^0.

These estimates are represented by their corresponding 2σ uncertainty ellipsoids. The combined estimate $\mathbf{c}$ using CI and the method ignoring the correlation have been shown. The dashed ellipsoids (green) are the calculated $\mathbf{P}_{cc}^*$ estimates using different values of the cross-correlation matrix (obtained using equal sampling) defined by $\tilde{\mathbf{P}}_{ab} = \rho \cdot \mathbf{I}$ as described in Subsection IV-B. For those values of the cross-correlation $\tilde{\mathbf{P}}_{ab}$ in which the exact optimal value $\mathbf{P}_{cc}^*$ is enclosed by the ellipsoid defined by $\mathbf{P}_{cc}^0$, it is safe to ignore the cross-correlation and still be consistent. However, if the ellipsoid representing the optimal $\mathbf{P}_{cc}^*$ encloses the $\mathbf{P}_{cc}^0$ ellipsoid then ignoring the correlation generates an inconsistent estimate. The CI algorithm achieves a consistent, yet conservative estimate.

VI. Conclusions

This paper analysed the consistency and applicability of three notable fusion algorithms for combining correlated random variables. It was shown that, although ignoring the non-zero correlation can cause inconsistency in the general case, there are cases where the consistency of the combined estimate can be achieved by simply neglecting the correlation. We derived conditions on the correlation under which one may simply ignore the correlation (as if it were zero) and apply an optimal fusion algorithm. Such conditions were given in the one-dimensional case and in a special case of high-dimensional estimation. This method of fusion

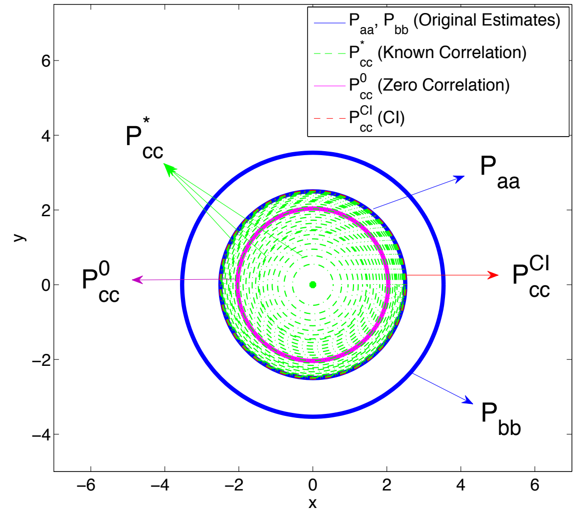

Fig. 2. Comparison of the obtained estimate $\mathbf{c}$ resulting from fusing 2-D estimates $\mathbf{a}$ and $\mathbf{b}$ using different fusion techniques. 2σ uncertainty bounds have been shown using the covariance ellipsoids (circles here).

will be considerably less conservative than covariance intersection.

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