Characterization of Weakly Convex Sets in Projective Space

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1 Convex Sets in $\mathcal{P}^n$

We start with a few simple observations about projective space $\mathcal{P}^n$. This space is formed from the n-sphere S^n by identifying opposite points. Thus, S^n is a 2-fold covering space of $\mathcal{P}^n$, and as a direct consequence, $\pi_1(\mathcal{P}^n) \approx Z_2$ for $n > 1$, since the n-sphere is simply connected for $n > 1$.

For $n = 1$, $\mathcal{P}^1$ and S^1 are topologically and geometrically identical, but we may still consider the 2-fold covering map of $\mathcal{P}^1 \rightarrow S^1$. Since S^1 and $\mathcal{P}^1$ are simple and easily understood, we will mainly be interested in the case $n > 1$, and henceforth, we restrict our attention to this case.

One may consider a metric structure on $\mathcal{P}^n$ and refer to lines and planes as geodesics, but this will not be useful in this note. We will only be interested in the projective geometric properties of $\mathcal{P}^n$ and not any metric properties. The most important objects are points, lines and planes, where we use the word plane to mean a hyper-plane of dimension $(n-1)$ in $\mathcal{P}^n$. Important properties are as follows.

1. Any two points in $\mathcal{P}^n$ lie on a unique line. The line is separated by the two points into two parts, called line segments.

2. A plane in $\mathcal{P}^n$ is projectively equivalent to $\mathcal{P}^{n-1}$.

3. A plane and a line not lying in the plane meet in exactly one point.

4. A plane in $\mathcal{P}^n$ is non-separating. If π is a plane in $\mathcal{P}^n$, then $\mathcal{P}^n - \pi$ is projectively equivalent to $\mathbb{R}^n$, lines and planes in $\mathcal{P}^n - \pi$ corresponding to lines and planes in $\mathbb{R}^n$.

It will be our practice to draw diagrams of configurations in projective space $\mathcal{P}^2$ or more generally $\mathcal{P}^n$ by selecting a plane π, thought of as the plane at infinity, and drawing objects in $\mathbb{R}^2$ or $\mathbb{R}^n$ by representing lines in $\mathcal{P}^n$ by lines in $\mathbb{R}^n$. It should be remembered that a line between two points is made up of two line segments, one which joins the points in $\mathbb{R}^n$ and the other which crosses the plane at infinity.

By a *convex set* in $\mathcal{P}^n$, we mean a *weakly* convex set. That is, a convex set S satisfies the condition that any two points X_1 and X_2 in S are joined by exactly one line segment lying in S. We assume as a matter of notational convenience that the empty set is a convex set.

The main purpose of this note is to show that if S is a convex set in $\mathcal{P}^n$, then there exists a plane in $\mathcal{P}^n$ that does not meet S. In this case, there is a one-to-one correspondence between convex sets in $\mathbb{R}^n$ and convex sets in $\mathcal{P}^n$.

2 Triangles in $\mathcal{P}^2$

In this section, we restrict attention to the 2-dimensional projective plane $\mathcal{P}^2$. A triangle in $\mathcal{P}^2$ consists of three non-collinear points along with geodesic segments joining the three pairs. Just to emphasize this, the word triangle is used to represent the three edges and the vertices, not a region bounded by the triangle. Three points in a weakly convex set S define a unique triangle lying in the set, since each pair of vertices is connected by a line segment. The perimeter of the triangle is the union of its three edges. We will consider two types of triangles. A triangle is called a *small triangle* if its perimeter is a null-homotopic path in $\mathcal{P}^2$. Otherwise, it is called a *large triangle.* It is easily seen that a triangle is small if and only if there exists a line that intersects the perimeter of the triangle in an even number of points. In making this observation, we exclude lines that contain any of the vertices of the triangle.

Small and large triangles are shown in fig 1

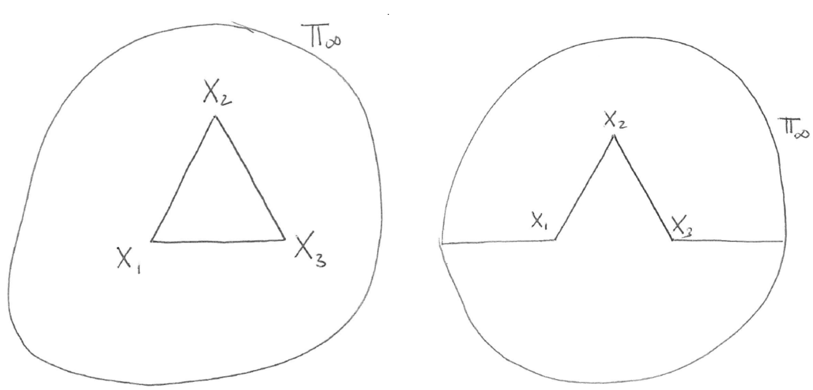

Figure 1: *Small (left) and large (right) triangles.*

One can be more specific, as the following lemma shows.

Lemma 2.1. *If T is small triangle in $\mathcal{P}^2$, then there is a line that does not intersect T in any point. If T is a large triangle, then there exists a line that meets it in one point, interior to one of the edges.*

Proof. First a definition: if g is a line segment joining two points in $\mathcal{P}^2$, then the *complementary* line segment is the other part of the complete line containing the two points. We include the two end points as part of the complementary line segment.

Consider a small triangle T and a line L that does not meet any of the three vertices. If L and T do not intersect, then we are done. The alternative is that it meets two of the sides g_1 and g_2. Let $\bar{g}_1$ and $\bar{g}_2$ be the line segments complementary to g_1 and g_2. Now, select two points $x_1 \in \bar{g}_1$ and $x_2 \in \bar{g}_2$, not vertices of the triangle, and let L be the line that joins these points. This line does not meet g_1 or g_2, and since it must meet the triangle in an even number of points, it does not meet g_2 either. Hence it does not meet the triangle.

If T is a small triangle, select a point x_1 in $\bar{g}_1$ and point x_2 in g_2, and join them with a line L. Since this line must meet the triangle in an odd number of points, it cannot meet g_3, and the proof is finished.

This lemma shows that the forms of the triangles shown in fig 1 represent the most general form of small and large triangles in $\mathcal{P}^2$, where the line guaranteed by lemma 2.1 is chosen as the line at infinity, π_∞.

2.1 Small triangles

We list some important facts about small triangles. Choosing a line not meeting a small triangle as the line at infinity in $\mathcal{P}^2$, one may envisage a small triangle as lying in the Euclidean plane $\mathbb{R}^2$. From this, several properties follow.

Lemma 2.2. *1. A small triangle separates $\mathcal{P}^2$ into two regions. One region is a topological disk, and the other one (containing the line at infinity) is not. The region homeomorphic to a disk is termed the* interior *of the triangle. The other region (homeomorphic to a projective plane, less a disk[1]) is the* exterior *of the triangle.*

2. If a small triangle in $\mathcal{P}^2$ is contained in a convex set S, then the interior of the triangle lies in S.

Proof. The first statement follows from the properties of a triangle in $\mathbb{R}^2$. To prove the second statement, let x be a point in the interior of the triangle, and suppose that $x \notin S$. Let L be any line through x. This line must meet the triangle at two points where it crosses the triangle. These points divide L into two segments g in the interior of the triangle, and $\bar{g}$ in the exterior of the triangle. One of these two segments must lie entirely inside S, since S is convex. However, since x is not in S, this cannot be the segment g, and so $\bar{g}$ lies in S. As line L rotates around the point x, the line segment $\bar{g}$ exterior to the triangle sweeps out the whole exterior of the triangle, which must therefore be contained in S. However, the exterior of the triangle contains the whole line at infinity π_∞, which contradicts the fact that S is convex (it cannot contain a complete line). From this contradiction, it follows that $x \in S$, and so the complete interior of the triangle lies in S.

[1] By a classical result, this is not a disk, see [1]

2.2 Large triangles

Now, we consider large triangles. It will be our goal to show that large triangles in a convex set S do not exist. However, this will need to be proved in several steps.

Notation. Let X_i be points in $\mathcal{P}^2$, not lying on a given plane π. A notation such as $[X_1 X_2 \ldots X_m]$ denotes the path consisting of line segments between successive points X_1 to X_m. Since there are two line segments between any two points, we separate two points X_i and X_{i+1} by a vertical bar $|$ to indicate that the line segment that crosses the plane π is to be chosen. For instance, the path $[X_1 X_2 X_3 | X_1]$ is a large triangle in which the segment $[X_3 | X_1]$ crosses π. A triangle will be large if and only if it contains an odd number of $|$ in its representation.

Notation: The notation $[X_1 X_2 X_3]_S$ denotes the triangle with vertices X_i and edges lying in S. In this notation, we do not use bars $|$ to indicate whether one of the edges crosses the plane π_∞.

Since it is easy to find a plane that misses one given side of a triangle, we see that a triangle is large if and only if there is some plane that meets just one of the edges.

Definition 2.3. An *incomplete solid triangle* in a set $S \subset \mathcal{P}^2$ is a small triangle that bounds an open region in S and such that exactly two edges of the triangle lie entirely in S.

In this definition, note that if S is a convex set then the complement of the edge not in S must lie in S. Next we prove the following lemma.

Lemma 2.4. *If a convex set S in $\mathcal{P}^2$ contains a large triangle, then it contains an incomplete solid triangle.*

Proof. Refer to fig 2 and fig 3 for an illustration of the following proof.

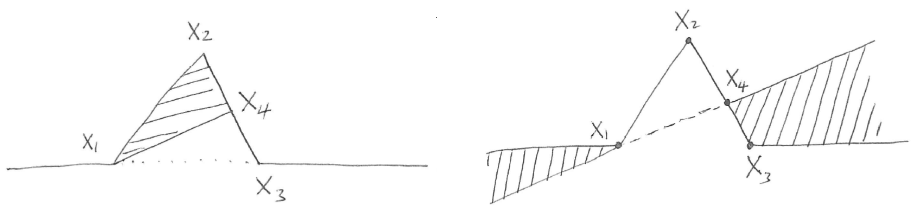

Figure 2: **Left:** *If $[X_1 X_4]$ is in S, then the triangle $[X_1 X_2 X_4 X_1]$ lies in S.* **Right:** *If $[X_1 | X_4]$ is in S, then the triangle $[X_1 | X_4 X_3 | X_1]$ lies in S.*

Let $[X_1 X_2 X_3 | X_1]$ be a large triangle contained in S. Consider a point $X(t)$ moving at constant speed along the line segment from X_2 to X_3, parametrized by $t \in [0, 1]$. Since $X \in S$, there is line segment from X_1 to $X(t)$ lying in S. Some of these line segments will cross the plane π and some will not. Let I_0 be

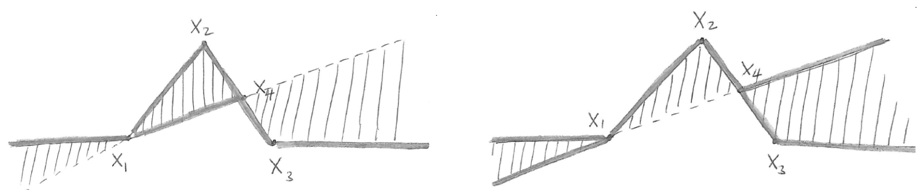

Figure 3: X_4 *is the point on* $[X_2X_3]$ *that separates small from large triangles* $[X_1X_2X_4]_S$. **Left:** *If* $[X_1X_4]$ *is in* S*, then the triangle* $[X_1|X_4X_3|X_1]$ *is an incomplete solid triangle.* **Right:** *If* $[X_1|X_4]$ *is in* S*, then the triangle* $[X_1X_2X_4X_1]$ *is an incomplete solid triangle.*

the set of values t such that the line segment from X_1 to $X(t)$ does not cross the plane π and let I_1 be the set of t such that the line segment from X_1 to $X(t)$ does cross the plane π. This may alternatively be stated as follows: I_0 is the set of t such that the triangle $[X_1X_2X(t)]_S$ is small, and I_1 is the set of t such that this triangle is large. Now $0 \in I_0$ since the edge $[X_1X_2] = [X_1X(0)]$ does not cross π. Similarly, $1 \in I_1$. However, each value of t must be in I_0 or I_1, so $\sup I_0 \geq \inf I_1$.

We make various observations.

1. If X_4 lies on the line segment $[X_2X_3]$, then
   (a) If $[X_1X_4]$ lies in S, then the triangle $[X_1X_2X_4X_1]$ bounds a region lying in S. (See fig 2 left).
   (b) If $[X_1|X_4]$ lies in S, then the triangle $[X_1|X_4X_3|X_1]$ bounds a region lying in S. (See fig 2 right).
2. If $X_4 = X(t)$ where $t < \sup I_0$, then $[X_1X_2X_4X_1]$ is a triangle with interior lying in S.
3. If $X_4 = X(t)$ where $t > \inf I_1$, then $[X_1|X_4X_3|X_1]$ is a triangle with interior lying in S.

If $\sup I_0 > \inf I_1$, then there exists a value of t such that $\inf I_1 < t < \sup I_0$. If $X_4 = X(t)$, then from the remarks above, both the segments $[X_4X_1]$ and $[X_1|X_4]$ lie in S, which contradicts the assumption that S is weakly convex.

The other possibilty is that $\sup I_0 = \inf I_0 = t$ for some t. We define $X_4 = X(t)$. There are then two cases.

1. If $[X_1X_4]$ is in S, then the triangle $[X_1|X_4X3|X_1]$ is an incomplete solid triangle. (See fig 3 left).
2. If on the other hand $[X_1X_4]$ is not in S, then the triangle $[X_1X_2X_4X_1]$ is an incomplete solid triangle. (See fig 3 right).

The following lemma implies that a convex set does not contain incomplete solid triangles.

Lemma 2.5. *Suppose a set S has the property that every two points are connected by at least one line segment. If S contains an incomplete solid triangle, then it contains a complete line.*

Proof. For this proof, refer to fig 4.

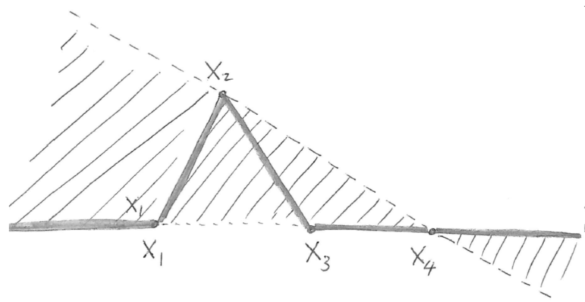

Figure 4: *The existence of the incomplete triangle $[X_1X_2X_3X_1]$ in a convex region S implies that a complete region lying between two lines lies in S, and hence, S contains a complete line.*

Suppose that that the vertices of an incomplete solid triangle are X_1, X_2 and X_3. We may assume that there is a plane π not meeting the triangle, or its interior lying in S. Let the edge $[X_1X_3]$ not lie entirely in S. Then $[X_1|X_3]$ does lie in S. By an argument similar to the one used previously, there exists a point X_4 on the line segment $[X_3|X_1]$ that divides the triangles $[X_2X_3X_4]_S$ that are small from those that are large. Note that if $[X_2X_3X_4]_S$ is large, then $[X_2X_1X_4]_S$ is small. Without loss of generality (for notational convenience only) we assume that the segment $[X_3X_4]$, not crossing the plane π, lies in S. It follows as before that the two triangles $[X_2X_3X_4X_2]$ and $[X_1|X_4|X_2X_1]$ are small and bound regions in S.

The union of the three triangles $[X_1X_2X_3X_1]$, $[X_2X_3X_4X_2]$ and $[X_1|X_4|X_2X_1]$ now consists of the complete region lying between the two lines $[X_4|X_2X_4]$ and $[X_4|X_1X_3X_4]$, and the interior of this region lies entirely in S. Finally, since X_4 also lies in S, any complete line lying strictly between these two lines will lie completely in S. This completes the proof of lemma 2.5. As a result we obtain the following result.

Lemma 2.6. *A convex set S can not contain a large triangle.*

3 Convex sets and planes

We consider planes in projective spaces. By a plane, we mean an $(n-1)$-dimensional plane in $\mathcal{P}^n$. In the case where $n = 2$, a plane is the same as a line, but we will continue to refer to them as planes when appropriate.

Lemma 3.7. *1. n distinct points $X_1, \ldots, X_n$ define a plane.*

2. Let X_1 and X_2 be two points lying in a plane. Then the line containing X_1 and X_2 must lie in the same plane.

Now, we consider the relationship between a convex set and a plane.

Lemma 3.8. *Let S be a convex set and π a plane. The intersection of S and π is a convex set. Further, $S - \pi$ is made up of two disjoint sets, $S - \pi = A \cup B$ where A and B are both convex.*

Proof. It is easy to see that $S \cap \pi$ is convex, since if X_1 and X_2 are both in $S \cap \pi$, then they are joined by a single line segment lying in S, and this must also lie in π according to lemma 3.7.

We now consider a relation defined on $S - \pi$. Two distinct points X_1 and X_2 are related (we write $X_1 \sim X_2$) if the line segment $[X_1 X_2]$ (not meeting π) lies in S. In addition, we define $X_1 \sim X_1$ for any point X_1. We now proceed to show that this is an equivalence relationship. The only thing we need to show is transitivity. Thus, suppose $X_1 \sim X_2$ and $X_2 \sim X_3$, but $X_1 \not\sim X_3$. In this case, the triangle $[X_1 X_2 X_3 | X_1]$ is a large triangle in S, which is not possible, according to lemma 2.6.

This equivalence relationship divides $S - \pi$ into two disjoint equivalent classes, A and B which are both convex, according to the definition of the relation $\sim$, and which are connected only by line segments crossing the plane π. This completes the proof.

4 The main separation theorem

The main result we want to achieve is as follows.

Theorem 4.9. *If S is a convex set in $\mathcal{P}^n$, then there exists a plane π in $\mathcal{P}^n$ that does not meet S.*

This theorem will be proved by induction on n. The case $n = 1$ is trivial. Note that a "plane" in $\mathcal{P}^1$ consists of a single point, so this is saying that if S is convex in $\mathcal{P}^1$, then there exists a point not in S.

Now, assume the theorem is true for $\mathcal{P}^{n-1}$, and consider $\mathcal{P}^n$. Let π_∞ be a plane in $\mathcal{P}^n$. If π_∞ does not meet S, then we are done. Otherwise, π_∞ meets S in a convex set, and since π_∞ is an instance of $\mathcal{P}^{n-1}$, there exists a plane μ_∞ (of dimension $n-2$) in π_∞ that is disjoint from S. Note that μ_∞ has codimension 1 in π_∞, or codimension 2 in $\mathcal{P}^n$. This situation is shown in fig 5, which will be used to illustrate the proof.

The plane π_∞ divides S into two convex sets A and B in $\mathbb{R}^n$. We may therefore find a further plane π_1 in $\mathbb{R}^n$ that separates A and B. This plane extends to a plane in $\mathcal{P}^n$, which may or may not meet S on π_∞. If it does not, then π_1 is the plane we are looking for, and the proof is complete. So, assume that $\pi_1 \cap \pi_\infty$ contains points of S.

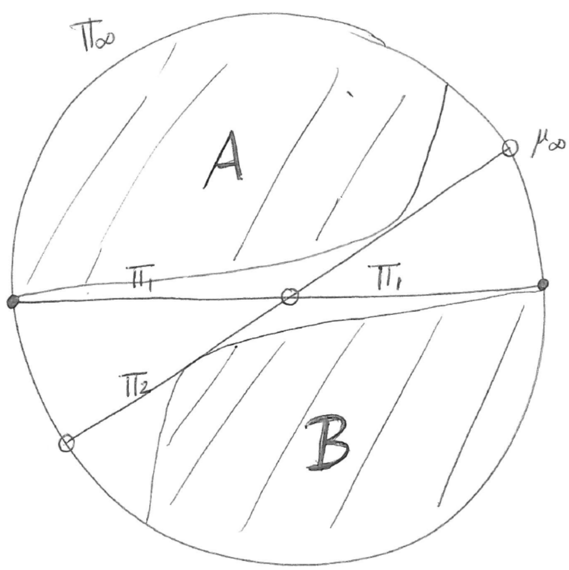

Figure 5: $\mathcal{P}^n$ *is separated by the planes* π_∞*,* π_1 *and* π_2 *into* 4 *regions. Any plane lying between* π_1 *and* π_2 *will be disjoint from the set* S*.*

Now, we consider the set of all planes in $\mathcal{P}^n$ that contain the codimension-2 plane μ_∞. There is a one-dimensional family of such planes, in fact they may be parametrized by S^1. Since π_∞ is one such plane, the remaining planes may be parametrized by the interval $(0, 1)$. We denote them by π_t where $t \in (0, 1)$. The intersection of π_t and π_∞ consists precisely of μ_∞, which does not contain points of S. Therefore, if any π_t is disjoint from A and B, then it is the plane we are looking for. On the other hand, π_t can not intersect both A and B, for if $X_1 \in A$ and $X_2 \in B$, then there is a line segment in S joining X_1 to X_2. This line segment must lie in π_t, since X_1 and X_2 do, but it must also cross π_∞, since A and B are only joined by a line segment passing through π_∞. However, $\pi_t \cap \pi_\infty = \mu_\infty$ which is disjoint from S, which supplies a contradiction.

Therefore, we suppose that every plane π_t meets exactly one of A and B. Let I_A be the set of those t for which π_t meets A and I_B be the set of t such that π_t meets B. It is easy to see that I_A and I_B are connected sets in $(0, 1)$ since if two planes π_{t_1} and π_{t_2} both meet A, then so does any plane π_t with $t_1 < t < t_2$. Therefore, there exists $s \in (0, 1)$ and a plane $\pi_2 = \pi_s$ such that π_t meets A for $t \in (0, s)$ and π_t meets B for $t \in (s, 1)$. Further, π_s itself meets either A or B.

Now, consider the two planes π_1 and π_2. They are not the same, because π_2 meets π_∞ in μ_∞ which does not contain a point in S, whereas $\pi_1 \cap \pi_\infty$ does contain a point in S. The three planes π_∞, π_1 and π_2 divide up $\mathcal{P}^n$ into 4 open regions as shown in fig 5. Both the planes π_1 and π_2 separate A from B. It follows that A and B are contained in only two of the four regions, and they must be opposite regions, as shown in fig 5.

To complete the proof, we consider a plane that lies between π_1 and π_2, passing through the regions of $\mathcal{P}^n - \pi_\infty$ that do not contain A or B. This plane will not meet S in $\mathcal{P}^n - \pi_\infty$. In addition, there does not exist a point lying on π_∞ between the planes π_1 and π_2 either, since such a point can not be joined to any point in A or B without crossing one of the planes π_1 and π_2 at a point not in π_∞.

This completes the proof.

References

[1] W. Massey. *Algebraic topology: An introduction.* Springer, 1977.