Gradient-like observers on semidirect products

C. Lageman, J. Trumpf and R. Mahony

Abstract— **This paper proposes a simple full state observer design for invariant state space systems where the state is evolving on a semidirect product of connected, finite-dimensional Lie groups. The design is based on a pair of cost functions defined on the Lie groups and consists of a copy of the observed system and a gradient-like innovation term. Under mild conditions the observer displays almost global exponential convergence. We illustrate the construction by an application in pose estimation.**

I. INTRODUCTION

Driven by the need for highly robust state estimation algorithms for autonomous robotic unmanned aerial, ground or submersible vehicles, there has been strong interest recently in nonlinear observer designs for systems whose state evolves on a Lie group. In connection with pose estimation for autonomous robotic vehicles the Lie group in question is the Special Euclidean group $SE(3)$, whose elements comprise of a rotation, describing the attitude of the vehicle, and a translation, describing its linear position. Thus, the group $SE(3)$ is the semidirect product of the Special Orthogonal group $SO(3)$ of rotations and $\mathbb{R}^3$.

Nonlinear observer design for such applications offers the potential of computationally simple state-estimation algorithms with strong robustness and global stability guarantees; as compared to the alternative of nonlinear filter designs (eg. extended Kalman filters [1] or particle filters [2]) that provide more information (*posterior* distributions for the state estimates) but usually require significant computational resources and rarely have strong global stability or robustness guarantees.

In the early nineties, Salcudean [3] proposed a nonlinear observer for the attitude estimation of a rigid-body using the unit quaternion representation of the Special Orthogonal group $SO(3)$. This work is seminal to a series of papers that develop nonlinear attitude observers for rigid-body dynamics [4], [5], [6], [7], [8], [9], [10] that exploit either the unit quaternion group structure or the rotation matrix Lie-group structure of $SO(3)$. The resulting attitude observers have comparable performance to state-of-the-art nonlinear filtering techniques [11], generally have much stronger global stability and robustness properties, and are simple to implement. The full pose estimation problem on the Special Euclidean group $SE(3)$ has also attracted recent attention [12], [13], [14], [15]. Theoretical work in this direction is less advanced due to the more complex algebraic and geometric structure of $SE(3)$. Recently, several authors have started to develop a theoretical foundation for observer design for systems with invariance properties, and in particular systems with a Lie-group state space and the natural left invariant dynamics [16], [17]. Early work in this direction considered uniform invariance properties across the system, measurements and observer design. More recent work recognises that it may be necessary to consider different invariance properties for the system than the measurements in order to obtain well conditioned observers [18], [19]. Results in this area are very recent and there is no well established observer design methodology or even analysis framework for invariant systems on a Lie group, even in the case of full state measurements.

In this paper, we study the design of full state observers for state space systems where the state is evolving on a semidirect product of connected, finite-dimensional Lie groups. We consider the case where full measurements are available for the system kinematics and propose a simple observer design based on a pair of cost functions measuring the size of the instantaneous observation error. By prescribing gradient dynamics for the observation error we arrive at a design consisting of a copy of the observed system and a gradient-like innovation term. This design is reminiscent of traditional full state observers and filters for linear systems evolving on vector spaces [20], [21]. Under mild assumptions, we prove almost global exponential convergence of the resulting observer and demonstrate its utility for practical pose observer design.

Following the introduction, Section II gives an overview of the notation used and the class of systems considered in this paper. In Section III we introduce our notion of error functions and cost functions, followed by a discussion of the observer design procedure in Section IV. Section IV-A contains the main contribution of this paper, namely the observer construction for semidirect products. The application to pose estimation is discussed in Section IV-B. The paper finishes with some conclusions in Section V.

II. NOTATION AND SYSTEM SETUP

Let G be a finite dimensional, connected Lie group with Lie algebra $\mathfrak{g}$. We denote the identity element in G by e, and the left and right multiplication with an element $X \in G$ by L_X and R_X, respectively. We use the representation of the tangent bundle of G by left or right translations of the Lie algebra, i.e. $T_eL_X\mathfrak{g}$ or $T_eR_X\mathfrak{g}$ for the tangent space T_XG

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of G at X. We use the simplified notation Xv for vectors $T_e L_X v \in T_X G$ and vX for vectors $T_e R_X v \in T_X G$ with $v \in \mathfrak{g}$. Furthermore, we assume that there is a Riemannian metric $\langle \cdot, \cdot \rangle$ on G. We do not make any invariance assumptions on this metric yet.

Consider a left invariant system on G of the form

$$\dot{X} = Xu, \tag{1}$$

where $u\colon \mathbb{R} \to \mathfrak{g}$ is a smooth function. Here, we call an input u of system (1) *admissible* if corresponding solutions of the system exist for all initial values and all initial times, and these solutions are unique, continuously differentiable and exist for all $t \in \mathbb{R}$.

This paper discusses observer design for a situation where we have (potentially noisy) measurements of X and u and want to build an observer that estimates X.

Example 1: As a first example we consider the special orthogonal group $\mathrm{SO}(3)$. We choose the representation of $\mathrm{SO}(3)$ by real, orthogonal 3×3 matrices, denoted by R or S. Its Lie algebra $\mathfrak{so}(3)$ is given by the real, skew-symmetric 3×3 matrices, denoted by Ω. The tangent spaces $T_R \,\mathrm{SO}(3)$ are identified with

$$\{R\Omega \mid \Omega \in \mathfrak{so}(3)\} \subset \mathbb{R}^{3\times 3}.$$

We equip $\mathrm{SO}(3)$ with the Riemannian metric induced by the Euclidean one on $\mathbb{R}^{3\times 3}$, i.e. $\langle A, B \rangle = \mathrm{tr}(A^\top B)$. On $\mathrm{SO}(3)$ we have the system

$$\dot{R} = R\Omega,$$

with $R\Omega$ denoting the matrix product, and $\Omega\colon \mathbb{R} \to \mathfrak{so}(3)$ admissible. This system models the kinematics of the attitude R of a coordinate frame fixed to a rigid body in 3D-space relative to an inertial frame. Here, Ω encodes the angular velocity measured in the body-fixed frame.

In applications, measurements of R are for example provided by a vision system, while measurements of Ω are obtained from on board gyrometers.

Example 2: The above example can easily be extended to model the full pose, including the position p in 3D-space, by moving to the special Euclidean group $\mathrm{SE}(3)$. It is well-known that $\mathrm{SE}(3)$ is the semidirect product of $\mathrm{SO}(3)$ by $\mathbb{R}^3$ via the map $\sigma(R)(p) = Rp$ for all $R \in \mathrm{SO}(3)$ and $p \in \mathbb{R}^3$ [22]. This yields the representation of $\mathrm{SE}(3)$ as

$$\{(R, p) \mid R \in \mathrm{SO}(3), p \in \mathbb{R}^3\},$$

where $\mathrm{SO}(3)$ is identified with the orthogonal 3×3 matrices as above. The group product is given by

$$(R, p)(S, q) = (RS, p + Rq).$$

On $\mathrm{SE}(3)$ we consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} R(t) & p(t) \end{pmatrix} = \begin{pmatrix} R & p \end{pmatrix}\begin{pmatrix} \Omega & V \end{pmatrix} = \begin{pmatrix} R\Omega & RV \end{pmatrix}$$

with $\Omega\colon \mathbb{R} \to \mathfrak{so}(3)$ and $V\colon \mathbb{R} \to \mathbb{R}^3$ admissible. Here, V is the linear velocity measured in the body-fixed frame.

In applications, measurements of p may again be provided by a vision system, while measurements of V are obtained by integrating measurements from on board linear accelerometers, possibly fused with approximative derivatives of the position measurements.

Our observer design aims to obtain autonomous dynamics of a specific *error function*. In fact, our construction yields gradient dynamics of this error with respect to a *cost function*. In the next section we briefly discuss the error and cost functions. For more details see [19] or [23].

III. Error functions and cost functions

Consider a pair of systems on G

$$\dot{X} = F_X(X, u, t), \tag{2}$$

$$\dot{\hat{X}} = F_{\hat{X}}(\hat{X}, Y, w, t) \tag{3}$$

with $F_X : G \times \mathfrak{g} \times \mathbb{R} \to TG$ and $F_{\hat{X}} : G \times G \times \mathfrak{g} \times \mathbb{R} \to TG$, $F_X(X, u, t) \in T_X G$ and $F_{\hat{X}}(\hat{X}, Y, w, t) \in T_{\hat{X}} G$. We regard system (3) as an observer for system (2) by feeding Y with a measurement of X and w with a measurement of u, where u is admissible for system (2) and (Y, w) is admissible for system (3). An *error function* is a smooth function $E\colon G \times G \to M$, where M is a smooth manifold.

Two particularly simple error functions on a Lie group G are the canonical *right invariant* error

$$E_r(X, \hat{X}) = \hat{X}X^{-1}$$

and the canonical *left invariant* error

$$E_l(X, \hat{X}) = X^{-1}\hat{X}.$$

Here, the label “invariant” refers to simultaneous state space transformations of both systems, e.g. we have $E_r(XS, \hat{X}S) = E_r(X, \hat{X})$ for all $X, \hat{X}, S \in G$. Note that for these canonical error functions $M = G$ and no error corresponds to the value e.

We need to be able to measure the size of an error value or, more generally, the distance between two elements of G. For this purpose we assume that we are given a smooth, non-negative *cost function* $f\colon G \times G \to \mathbb{R}$. Furthermore, let the diagonal $\Delta = \{(X, X) \mid X \in G\}$ consist of global minima of f. We can use f to measure the size of an error value, e.g. E_r, by computing $f(E_r, e)$. Note that the cost function f need not be related to the geometry of G, although one possible choice would be the geodesic distance.

Recall that the Riemannian gradient of f with respect to the product metric $\langle \cdot, \cdot \rangle_p$ on $G \times G$ is defined by

$$\langle \mathrm{grad}\, f(\hat{X}, Y), (\eta, \zeta) \rangle_p = df(\hat{X}, Y)(\eta, \zeta)$$

for all $Y, \hat{X} \in G$, $\eta \in T_{\hat{X}} G$, $\zeta \in T_Y G$. Since we use the product metric, the gradient splits into the gradients with respect to the first and second parameter, i.e.

$$\langle \mathrm{grad}\, f(\hat{X}, Y), (\eta, \zeta) \rangle_p = \langle \mathrm{grad}_1 f(\hat{X}, Y), \eta \rangle + \langle \mathrm{grad}_2 f(\hat{X}, Y), \zeta \rangle.$$

IV. Gradient-like observers on Lie groups

Our focus is on building observers with strong convergence and robustness properties. The idea behind our proposed observer design is hence to prescibe *gradient dynamics* for the instantaneous observer error and to deduce the observer design from that. Theorem 1 below is taken from [23] and contains the main result in this regard. It can be interpreted as an instance of an *internal model principle*, since the resulting design necessarily contains a copy of the observed system's dynamics. The result sharpens a general structure theorem for invariant observers due to Bonnabel et al. [17], [16]. While the resulting design is similar to the general design proposed there, our observer is not necessarily an invariant system (cf. also [18]).

Theorem 1: Let $f\colon G \times G \to \mathbb{R}$ be a smooth cost function. Consider the general observer (3) in the noise free case, i.e. $Y = X$ and $w = u$. Then we have gradient dynamics for the canonical right invariant error E_r, i.e.

$$\dot{E}_r = -\operatorname{grad}_1 f(E_r, e), \tag{4}$$

if and only if

$$F_{\hat{X}}(\hat{X}, Y, w, t) = \hat{X}w - T_{\hat{X}Y^{-1}} R_Y \operatorname{grad}_1 f(\hat{X}Y^{-1}, e).$$

Theorem 1 yields the gradient-like observer

$$\dot{\hat{X}} = \hat{X}w - T_{\hat{X}Y^{-1}} R_Y \operatorname{grad}_1 f(\hat{X}Y^{-1}, e) \tag{5}$$

We immediately get the following strong convergence result.

Theorem 2: Assume that $Y \mapsto f(Y, e)$ is a Morse-Bott function with a global minimum at e and no other local minima. Both canonical errors (E_r and E_l) of the observer (5) converge to e for generic initial conditions. Furthermore, the convergence of E_r is locally exponential near e.

Proof: The right invariant error E_r has the gradient dynamics (4). Standard results for the gradient dynamics of Morse-Bott functions [24] yield the the generic convergence to the global minimum and the local exponential convergence. ■

Remark 1: Consider the left invariant system (1) and noisy measurements $w = u + \delta$ and $Y = XN_r$, with additive driving noise $\delta \in \mathfrak{g}$ and right multiplicative state noise $N_r \in G$. A straightforward calculation yields

$$\dot{E}_r = \operatorname{Ad}_{\hat{X}} \delta E_r - T_{E_r N_r^{-1}} R_{X^{-1} N_r X} \operatorname{grad}_1 f(E_r N_r^{-1}, e)$$

for the canonical right invariant error of the observer (5). A suitably bounded noise will yield at least a practical stability result in these cases.

A. Observers on semidirect products

Since it is of importance for practical applications, we will now consider the special case of gradient-like observers on the semidirect product of two Lie groups. Let us recall some well-known facts on semidirect products, for detailed information we refer the reader to standard literature like [22]. The semidirect product $G_1 \ltimes G_2$ of two Lie groups G_1 and G_2 is the cartesian product of the groups as manifolds, with the multiplication given by $(g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 \phi(g_1)(h_2))$, where $\phi\colon G_1 \to \operatorname{Aut}(G_2)$ is a smooth homomorphism into the automorphism group of G_2. In the following, $\psi\colon G_2 \to \operatorname{Hom}(G_1, G_2)$ will be the map $\psi(g_2)(g_1) = \phi(g_1)(g_2)$. The inverse $(g_1, g_2)^{-1}$ is given by $(g_1^{-1}, \phi(g_1^{-1}) g_2^{-1})$. We will denote the components of an element of $G_1 \ltimes G_2$ by X_1 and X_2. In the remainder of this section, the group G will be the semidirect product of the Lie groups G_1 and G_2.

The left-invariant dynamics (1) have the following form on the semidirect product.

$$\begin{aligned} \dot{X}_1 &= X_1 u_1 \\ \dot{X}_2 &= X_2 T_e \phi(X_1) u_1 \end{aligned}$$

We consider the observer (5) for a cost which splits into a sum of independent costs on G_1 and G_2.

For our calculations we need the following formulas for the derivatives of the left and right multiplication that can be obtained by a straightforward application of calculus rules.

Lemma 1: For all $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in G$ and $(w_1, w_2) \in T_Y G$, we have

$$\begin{aligned} T_Y R_X(w_1, w_2) &= \big(T_{Y_1} R_{X_1} w_1, \\ &\quad T_{\phi(Y_1)(X_2)} L_{Y_2} T_{Y_1} \psi(X_2) w_1 + T_{Y_2} R_{\phi(Y_1)(X_2)} w_2\big), \\ T_Y L_X(w_1, w_2) &= \big(T_{Y_1} L_{X_1} w_1, \\ &\quad T_{\phi(X_1)(Y_2)} L_{X_2} T_{Y_2} \phi(X_1) w_2\big). \end{aligned}$$

Direct calculations now yield the following formulas for the canonical errors on a semidirect product.

Lemma 2: For all $X = (X_1, X_2)$ and $\hat{X} = (\hat{X}_1, \hat{X}_2) \in G$, we have

$$\begin{aligned} E_r &= \Big(\hat{X}_1 X_1^{-1}, \hat{X}_2 \phi(\hat{X}_1 X_1^{-1})(X_2^{-1})\Big), \\ E_l &= \Big(X_1^{-1} \hat{X}_1, \phi(X_1^{-1})(X_2^{-1} \hat{X}_2)\Big). \end{aligned}$$

In order to achieve a clearer presentation we will use the following Lemma in the proof of Theorem 3 below.

Lemma 3: If f and the Riemmanian metric are left invariant, then for all $X, Y, Z \in G$

$$T_X L_{Y^{-1}} \operatorname{grad}_1 f(X, YZ) = \operatorname{grad}_1 f(Y^{-1}X, Z)$$

If f and the Riemmanian metric are right invariant, then for all $X, Y, Z \in G$

$$T_X R_{Z^{-1}} \operatorname{grad}_1 f(X, YZ) = \operatorname{grad}_1 f(XZ^{-1}, Y)$$

Proof: If f is left invariant then $f \circ L_Y = f$. Using this fact and the standard rules for transformations of Riemannian gradients, we have that

$$\begin{aligned} \operatorname{grad}_1 f(Y^{-1}X, Z) &= \operatorname{grad}_1 (f \circ L_Y)(Y^{-1}X, Z) \\ &= (T_{Y^{-1}X} L_Y)^* \operatorname{grad}_1 f(X, YZ), \end{aligned}$$

where $(T_{Y^{-1}X} L_Y)^*$ denotes the Hilbert space adjoint of the linear map $T_{Y^{-1}X} L_Y$. Since the Riemannian metric is left invariant, we have that $(T_{Y^{-1}X} L_Y)^* = T_X L_{Y^{-1}}$. The right invariant case follows from an analogous argument. ■

The next theorem now shows the form of our gradient-like observers for the special case of a semidirect product of two Lie groups.

Theorem 3: Let $G = G_1 \ltimes G_2$ be a semidirect product of Lie groups. Assume that we have a cost function $f\colon G \times G \to \mathbb{R}$ such that $f(X,Y) = f_1(X_1,Y_1) + f_2(X_2,Y_2)$ with $f_1\colon G_1 \times G_1 \to \mathbb{R}$ and $f_2\colon G_2 \times G_2 \to \mathbb{R}$ cost functions on G_1 and G_2, respectively. Furthermore, assume that the Riemannian metric on G is the sum of Riemannian metrics on G_1 and G_2. If f_1, f_2 and the Riemannian metrics are right invariant then the observer (5) has the form

$$\begin{aligned}
\dot{\hat{X}}_1 &= \hat{X}_1 w_1 - \operatorname{grad}_1 f_1(\hat{X}_1, Y_1) \\
\dot{\hat{X}}_2 &= \hat{X}_2 T_e\phi(\hat{X}_1)(w_2) - T_{\phi(\hat{X}_1Y_1^{-1})(Y_2)} L_{\hat{X}_2\phi(\hat{X}_1Y_1^{-1})(Y_2^{-1})} \\
&\quad T_{\hat{X}_1Y_1^{-1}}\psi(Y_2)\operatorname{grad}_1 f_1(\hat{X}_1Y_1^{-1}, e) \\
&\quad - \operatorname{grad}_1 f_2(\hat{X}_2, \phi(\hat{X}_1Y_1^{-1})(Y_2))
\end{aligned}$$

Proof: We use the notation $(Z)_1$ and $(Z)_2$ for the G_1 and G_2 component of an expression, respectively. Applying Lemmas 1 and 3 to the observer equation (5), we see that

$$\begin{aligned}
\dot{\hat{X}}_1 &= \hat{X}_1 w_1 - T_{\hat{X}_1Y_1^{-1}} R_{Y_1} \operatorname{grad}_1 f_1(\hat{X}_1Y_1^{-1}, e) \\
&= \hat{X}_1 w_1 - \operatorname{grad}_1 f_1(\hat{X}_1, Y_1)
\end{aligned}$$

$$\begin{aligned}
\dot{\hat{X}}_2 &= \hat{X}_2 T_e\phi(\hat{X}_1) w_2 - T_{\phi((\hat{X}Y^{-1})_1)(Y_2)} L_{(\hat{X}Y^{-1})_2} \\
&\quad T_{(\hat{X}Y^{-1})_1}\psi(Y_2)\operatorname{grad}_1 f_1(\hat{X}_1Y_1^{-1}, e) \\
&\quad - T_{(\hat{X}Y^{-1})_2} R_{\phi((\hat{X}Y^{-1})_1)(Y_2)} \operatorname{grad}_1 f_2((\hat{X}Y^{-1})_2, e) \\
&= \hat{X}_2 T_e\phi(\hat{X}_1) w_2 - T_{\phi(\hat{X}_1Y_1^{-1})(Y_2)} L_{\hat{X}_2\phi(\hat{X}_1Y_1^{-1})(Y_2^{-1})} \\
&\quad T_{\hat{X}_1Y_1^{-1}}\psi(Y_2)\operatorname{grad}_1 f_1(\hat{X}_1Y_1^{-1}, e) \\
&\quad - \operatorname{grad}_1 f_2((\hat{X}Y^{-1})_2\phi((\hat{X}Y^{-1})_1)(Y_2), \\
&\qquad\qquad \phi((\hat{X}Y^{-1})_1)(Y_2)) \\
&= \hat{X}_2 T_e\phi(\hat{X}_1) w_2 - T_{\phi(\hat{X}_1Y_1^{-1})(Y_2)} L_{\hat{X}_2\phi(\hat{X}_1Y_1^{-1})(Y_2^{-1})} \\
&\quad T_{\hat{X}_1Y_1^{-1}}\psi(Y_2)\operatorname{grad}_1 f_1(\hat{X}_1Y_1^{-1}, e) \\
&\quad - \operatorname{grad}_1 f_2(\hat{X}_2\phi((\hat{X}Y^{-1})_1)(Y_2^{-1})\phi((\hat{X}Y^{-1})_1)(Y_2), \\
&\qquad\qquad \phi((\hat{X}Y^{-1})_1)(Y_2)) \\
&= \hat{X}_2 T_e\phi(\hat{X}_1) w_2 - T_{\phi(\hat{X}_1Y_1^{-1})(Y_2)} L_{\hat{X}_2\phi(\hat{X}_1Y_1^{-1})(Y_2^{-1})} \\
&\quad T_{\hat{X}_1Y_1^{-1}}\psi(Y_2)\operatorname{grad}_1 f_1(\hat{X}_1Y_1^{-1}, e) \\
&\quad - \operatorname{grad}_1 f_2(\hat{X}_2, \phi(\hat{X}_1Y_1^{-1})(Y_2))
\end{aligned}$$

■

B. Example: Pose estimation on SE(3)

As an example, consider the construction of an observer on the group SE(3), cf. Example 2 in Section II. To construct our gradient-like observers we equip SO(3) again with the Riemannian metric induced by the Euclidian one on $\mathbb{R}^{3\times3}$, $\langle A, B\rangle = \operatorname{tr}(A^TB)$. This metric is biinvariant. We can define the biinvariant cost function $f_1(\hat{R}, R) = \frac{k_R}{2}\|R - \hat{R}\|_F^2$ on SO(3), with $\|\cdot\|_F$ denoting the Frobenius norm. The gradient $\operatorname{grad}_1 f_1(\hat{R}, R)$ is $-k_R\hat{R}\mathbb{P}_{\mathfrak{so}(3)}(\hat{R}^TR)$, where $\mathbb{P}_{\mathfrak{so}(3)}(A)$ denotes the skew-symmetric part of a matrix A. The space $\mathbb{R}^3$ is equipped with the Euclidean metric and the cost function $f_2(\hat{p}, p) = \frac{k_p}{2}\|\hat{p} - p\|_E^2$, with $\|\cdot\|_E$ denoting the Euclidean norm. The cost f_2 and the Euclidean metric are obviously biinvariant under the additive action of $\mathbb{R}^3$ on itself. The gradient of f_2 is $\operatorname{grad}_1 f_2(\hat{p}, p) = k_p(\hat{p} - p)$. Applying Theorem 3 we get the following gradient-like observer on SE(3).

$$\begin{aligned}
\dot{\hat{R}} &= \hat{R}\Omega + k_R\hat{R}\mathbb{P}_{\mathfrak{so}(3)}(\hat{R}^TR) \\
\dot{\hat{p}} &= \hat{R}V + k_R\hat{R}R^T\mathbb{P}_{\mathfrak{so}(3)}(R\hat{R}^T)p - k_p(\hat{p} - \hat{R}R^Tp)
\end{aligned}$$

This is precisely the complementary filter presented in [14].

V. Conclusions

We have presented a simple full state observer design for invariant state space systems where the state evolves on a semidirect product of finite-dimensional, connected Lie groups. The resulting observer has gradient error dynamics and hence exhibits almost global exponential convergence under mild conditions on the underlying cost function. We have shown that in the case of the Special Euclidean Group SE(3) the resulting observer coincides with the complementary filter previously presented by the authors. That filter has previously been shown to yield excellent performance in a pose estimation scenario with inertial and vision based sensor measurements.

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