Generalised Rigidity and Path-Rigidity for Agent Formations

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Abstract— **The classical concept of *rigidity* characterises conditions under which distance constraints between agents in $\mathbb{R}^3$ enforce a rigid structure on the whole collection of agents. The present paper has two goals. Firstly, we propose a generalised theory for rigidity to allow heterogeneous agent states on non-Euclidean spaces and general non-linear relative state constraints. To do this, we characterise rigidity as an invariance property with respect to a topological group action that is introduced as a natural structure in the problem formulation. Secondly, we use this new framework to formulate a new concept of *path-rigidity*, which captures the property that allows a rigid formation to be steered continuously between any two configurations that are congruent. This is an important property for path planning and control of rigid formations. The main result of the second part of the paper provides a simple and easily checked condition to determine if a globally rigid formation is also path-rigid.**

I. Introduction

Over the past few decades, the classical concept of rigidity [8] involving distance constraints between nodes in $\mathbb{R}^2$ or $\mathbb{R}^3$ has drawn widespread interest for applications such as the design of beam structures or the study of chemical molecules (see e.g. the collection of references in [13]). Recently, more general notions of rigidity have been applied to problems of network localisation [5], computer-aided design (CAD) [10] and formation control [1].

In formation control, rigidity theory can not only be used to determine whether a desired formation is uniquely defined by the available sensor measurements, but it can also play a key role in the control analysis [1]. While numerous control architectures have been proposed for formations based on classical range-only constraints with states in $\mathbb{R}^2$ or $\mathbb{R}^3$ [9], [3], growing interest in the use of other sensor modalities and more general agent state-spaces has led to the need for a more general development of the concepts of rigidity. For example, formations based only on relative bearing or direction measurements [4], [6] will always be invariant to scale, and are therefore never rigid in the normal sense. To address this, the term *parallel rigidity* [4] has been specifically introduced. However, this ad-hoc definition is no longer valid for scenarios that combine direction and distance constraints [12]. A major limitation of the literature in dealing with such situations is the dependence of classical rigidity theory on agent states lying in Euclidean space, while common measurement modalities (e.g. bearings) depend on the full pose of agents. We note a recent exception that considers rigidity with regard to states in $\mathrm{SE}(2)$ [14].

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In [10], rigidity is interpreted as a type of symmetry action of a group. This formulation has been used to solve system constraints in Computer Aided Design (CAD) applications. A similar formulation has been used for the notion of *affine rigidity* [7]. Apart from these few references there appears to be little other work in this direction and to the best of the authors' knowledge, there is no general formation of rigidity based on invariance principles. In particular, the authors are not aware of any existing theory that allows the case of heterogeneous (non-Euclidean) agent states as is necessary in modern robotics applications with formations of different vehicles and multiple sensor modalities. Clearly, as autonomous vehicle sensor networks and agent models become more sophisticated, there is a strong motivation to generalise existing rigidity theory so that it may be more readily and more widely applied.

In this paper, we present a general framework for rigidity analysis based on invariance principles. Although we use formation control as a primary motivation, the proposed framework is general and will apply to a wide range of applications. We define rigidity in the context of an explicit symmetry action on the full state-space that preserves a specified set of state constraints. We define *generalised rigidity* by the condition that the *only* transformations admitted by the constraints are those associated with the natural symmetry. A particular benefit of the proposed framework is that neither the vehicles, nor the relative state constraints, need to lie in identical state-spaces. We develop generalised rigidity theory to the extent that we define and motivate the classical concepts of *local rigidity* and *global rigidity*. In the second part of the paper we propose a new concept of *path-rigidity*. This concept captures the particular case of a globally rigid system where any two congruent configurations are connected by a continuous path preserving all constraints, allowing the formation to be steered from one configuration to the other. The principal technical result is the derivation of an easily verifiable group theoretic characterisation of path-rigidity for a globally rigid system.

Following the present introduction, Section II formally introduces the system structure necessary for the framework and defines helpful terminology. In Section III we present the formal definition of generalised rigidity for formations, followed by some analysis and discussion with regard to *path-rigidity* in Section IV. Finally, we provide a brief conclusion in Section V.

II. Problem formulation

In this section we develop the framework with which we study rigidity, and introduce some key concepts for the discussion. The formulation applies to a very broad class of agent state-spaces and sensor modalities, with a topological group describing the symmetries of the system. The generality of the formulation is a significant consideration in this paper, and we demonstrate it with several examples. Subsection II-A models the full state-space and the available sensor measurements, along with associated assumptions. Subsection II-B then introduces the notions of symmetry with which we define rigidity.

A. Generalised agent networks and formations

Consider a system whose full state x lies in a topological space $\mathcal{M}^\tau := (\mathcal{M}, \tau(\mathcal{M}))$, where $\tau(\mathcal{M})$ denotes a Hausdorff topology on the set $\mathcal{M}$. We envision the full state x being derived from the individual states of n agents; however, we do not specify a particular construction of the full state-space $\mathcal{M}^\tau$ from the individual state-spaces, in order to preserve the generality of our formulation.

Example 2.1: A simple and very common scenario is where the ith agent state x_i lies in a Hausdorff topological space $\mathcal{M}_i^\tau$, and the full system state lies in the product space $\mathcal{M}^\wp := \prod_{i=1}^n \mathcal{M}_i^\tau$. In this case, $\mathcal{M}^\wp$ inherits the Hausdorff property from the individual state-spaces $\mathcal{M}_i^\tau$. Note that this construction does not assume the individual state-spaces to be identical, i.e. we do not require $\mathcal{M}_i^\tau = \mathcal{M}_j^\tau$ for $i \neq j$. ⋄

Example 2.2: To motivate the consideration of non-product structures for $\mathcal{M}^\tau$, suppose that all agent states $x_i \in \mathcal{M}_i^\tau$ lie in the same Hausdorff topological space (i.e. that $\mathcal{M}_i^\tau = \mathcal{M}_j^\tau$ for all $i, j \in \{1, \ldots, n\}$), and that the agents are regarded as *interchangeable* (i.e. that we are not concerned with which agent assumes each position). In this case, a state $\tilde{x} = (x_1, \ldots, x_n)$ in the product space $\mathcal{M}^\wp$ is equivalent to another state $\tilde{x}' \in \mathcal{M}^\wp$ if it can be obtained from $\tilde{x}'$ by switching the positions of the agents (or reassigning the agent indexes). Let $\mathbf{P}_n(\tilde{x})$ denote the group of permutations of the n-tuple $\tilde{x}$; that is, $\sigma(\tilde{x}) := (\sigma_1(\tilde{x}), \ldots, \sigma_n(\tilde{x}))$ for $\sigma \in \mathbf{P}_n(\tilde{x})$, with each $\sigma_i(\tilde{x}) = x_j$ for some $j \in \{1, \ldots, n\}$ and each j appearing exactly once in the list. With interchangeable agents, the full system state x lies in the quotient space $\mathcal{M}^\tau := \mathcal{M}^\wp / \sim$, where $\tilde{x} \sim \tilde{x}' \Leftrightarrow \exists \sigma \in \mathbf{P}_n(\tilde{x}) : \tilde{x}' = \sigma(\tilde{x})$ defines an equivalence relation for $\tilde{x}, \tilde{x}' \in \mathcal{M}^\wp$. The quotient topology $\tau(\mathcal{M})$ is the final topology of the natural projection $\pi : \mathcal{M}^\wp \to \mathcal{M}^\wp / \sim$; equivalently, a set $\mathcal{U} \subseteq \mathcal{M}^\tau$ is open if and only if the pre-image $\pi^{-1}(\mathcal{U}) \subseteq \mathcal{M}^\wp$ is open. Observe that the map π is open since the equivalence relation maps open sets to a union of open sets. Furthermore, the graph of the equivalence relation is closed in $\mathcal{M}^\wp \times \mathcal{M}^\wp$ since it is the product of $\mathcal{M}^\wp$ with a finite number of singletons corresponding to the permutations of the agents. These conditions ensure that the quotient space $\mathcal{M}^\tau$ is Hausdorff [11, Proposition 7.1.6]. ⋄

Example 2.3: A further practical consideration is the exclusion of some points from a product space $\mathcal{M}^\wp$. Consider a collection of physical vehicles in a common state-space, i.e. where $x_i \in \mathcal{M}_i^\tau$ and $\mathcal{M}_i^\tau = \mathcal{M}_j^\tau$ for all $i \neq j$. With this arrangement, the case where $x_i = x_j$ is not physically feasible since the vehicles cannot be physically co-located. Such points can be problematic for particular sensor modalities, e.g. direction measurements become ill-defined when the agents are co-located (see Example 2.10). For the purposes of rigidity analysis, it is therefore appropriate to consider the state-space $\mathcal{M}^\tau := \mathcal{M}^\wp \setminus \mathcal{W}^\tau$, where $\mathcal{W}^\tau := \{x \in \mathcal{M}^\wp \mid x_i = x_j \text{ for some } i \neq j\}$. Here, $\mathcal{M}^\tau$ is an open subset of $\mathcal{M}^\wp$, and has the induced topology of the product space; i.e., a set $\mathcal{U} \subseteq \mathcal{M}^\tau$ is open if and only if it is open as a subset $\mathcal{U} \subseteq \mathcal{M}^\wp$. We emphasise that $\mathcal{M}^\tau$ is *not* a product space; each vehicle can assume any state in $\mathcal{M}_i^\tau$, but its position imposes a constraint on those of the others. ⋄

The state of the system is measured by an output map $h : \mathcal{M}^\tau \to \mathcal{Y}$, which will be used to specify constraints on the system state. For the general notion of rigidity, we do not require a topology on the output space $\mathcal{Y}$. Should a topology $\tau(\mathcal{Y})$ be considered, we denote the topological space by $\mathcal{Y}^\tau$. We will use $y := h(x)$ to denote the particular output value of a state x. Throughout the following examples and remarks, we will commonly suppose $h(x)$ is composed of m individual *sensor modalities* as outlined in Remark 2.4.

Remark 2.4: The typical interpretation of this formulation is that $y = (y_1, \ldots, y_m)$ will be composed of m individual state measurements y_k, each of which corresponds to an available *sensor modality* $h_k : \mathcal{M}^\tau \to \mathcal{Y}_k$ with $\mathcal{Y}^\wp := \prod_{k=1}^m \mathcal{Y}_k$. Often, a sensor modality h_k will not be a function of the full state x, but will instead describe a relative measurement between only two agent states. We emphasise that the individual sensor modalities h_k or the output spaces $\mathcal{Y}_k$ need not be the same, and we will also consider cases where $\mathcal{Y}$ does not possess a product structure. ⋄

Remark 2.5: Although a topology on $\mathcal{Y}$ (or $\mathcal{Y}_k$) is often not strictly required for rigidity analysis, it is desirable for many applications that $h(x)$ be continuous (or even differentiable). For example, this may be the case when $h(x)$ is used to derive a control law for a group of vehicles. We will therefore provide some discussion concerning the continuity of h_k in the sequel, with the assumption of at least a T_0 (Kolmogorov) topology on $\mathcal{Y}$ (or each $\mathcal{Y}_k$). Note that some common sensor modalities (e.g. directions, see Example 2.10) are not continuous with any finer topology. The product space $\mathcal{Y}^\wp$ will inherit the T_0 property, and also preserve the continuity of each $h_k(x)$. ⋄

Remark 2.6: We emphasise that for this paper we are not interested in the availability of a measurement y_k to particular agents in the system. Rigidity is concerned only with whether certain constraints $y_k = y_k^\star$ are satisfied; the information topology of the network only becomes relevant when considering applications such as control algorithms. ⋄

Remark 2.7: Although we regard h_k as a sensor modality, it may alternatively be interpreted as a *task function* that measures the system state with respect to a goal. Examples of task functions are described in [2]. ⋄

Example 2.8: Consider two agents with states $x_i, x_j \in \mathbb{R}^d$ (for $d \geq 1$), and a product structure on the full state-space. A *range* or *distance* measurement between these agents is given by

$$y_k := h_k(x) := \|x_i - x_j\| \in \mathbb{R}_{\geq 0}, \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector and $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. ⋄

Example 2.9: Consider two agents with states[1] $X_i, X_j \in \mathrm{SE}(3)$ in the Special Euclidean group of dimension 3, and a product structure on the full state-space. The matrix representation of each state is

$$X_i := \begin{pmatrix} R_i & \xi_i \\ 0_3^\top & 1 \end{pmatrix},$$

where $R_i \in \mathrm{SO}(3)$ is a 3×3 rotation matrix (i.e. an orthogonal matrix of determinant 1), 0_3 denotes the 3×1 zero vector and $\xi_i \in \mathbb{R}^3$. Let $\mathring{y}_j \in \mathbb{R}^3$ be a point fixed with respect to the coordinate frame X_j (typically the origin). A *position* measurement of $\mathring{y}_j$ from agent i is described using a sensor modality $h_k : \mathrm{SE}(3) \times \mathrm{SE}(3) \to \mathbb{R}^3$ as follows:

$$\bar{y}_k := \overline{h_k(X)} := X_i X_j^{-1} \bar{\mathring{y}}_j. \tag{2}$$

Here, $\bar{v} := (v^\top, 1)^\top$ denotes a vector v expressed in homogeneous coordinates. The intuition here is that y_k expresses the position of $\mathring{y}_j$ with respect to the body-fixed frame X_i. ⋄

Example 2.10: Consider two agents with states $x_i, x_j \in \mathbb{R}^3$ and a product structure on the full state-space. An inertial *direction* measurement between agents i and j is defined by

$$y_k := h_k(x) := \begin{cases} \frac{(x_i - x_j)}{\|x_i - x_j\|} \in \mathbb{S}^2 & x_i \neq x_j \\ \aleph & \text{otherwise,} \end{cases} \tag{3}$$

where $\mathbb{S}^2$ denotes the unit sphere. The measurement space is $\mathcal{Y}_k := \{\mathbb{S}^2 \cup \{\aleph\}\}$, which we give the T_0 topology $\tau(\mathcal{Y}_k) := \{\tau(\mathbb{S}^2), \{\mathbb{S}^2 \cup \{\aleph\}\}\}$ to make $h_k(x)$ continuous. Note that this is the final topology with respect to $h_k(x)$, i.e. there is no T_1 topology such that h_k is continuous. ⋄

In some scenarios, a sensor may detect another vehicle without identifying which vehicle it is.

Example 2.11: Suppose agent p is equipped with a sensor modality $h_k(x)$ that measures the range to each other agent, without differentiating which agent corresponds to which distance. We model this sensor modality as a mapping

$$h_k(x) := \{\|x_i - x_p\|\}_{i \neq p} \in \mathcal{Y}_k^\tau, \tag{4}$$

where $\mathcal{Y}_k^\tau := \mathbb{R}_{\geq 0}^{n-1} / \sim_{y_k}$. Here, the equivalence relation is defined by $y_k \sim_{y_k} y_k' \Leftrightarrow \exists \sigma \in \mathbf{P}_{n-1}(y_k) : y_k' = \sigma(y_k)$. ⋄

We now provide a generalised definition of an agent network, based on the state $x \in \mathcal{M}^\tau$ and output $y \in \mathcal{Y}$.

Definition 2.12: A *generalised agent network* $\mathcal{N} := (\mathcal{M}^\tau, \mathcal{Y}, h)$ consists of a Hausdorff topological space $\mathcal{M}^\tau$, an output space $\mathcal{Y}$ and an output map $h : \mathcal{M}^\tau \to \mathcal{Y}$. ⋄

Although our generalised notion of an agent network does not involve a graph structure, it allows a similar interpretation to the classical concept of a network via the construction of $\mathcal{M}^\tau$ and h, as illustrated by the prior examples and remarks. For convenience, we will simply use the term *agent network* for this generalised form in the sequel. For the discussion of agent networks we introduce the following concepts.

Definition 2.13: A *configuration* of an agent network $\mathcal{N} := (\mathcal{M}^\tau, \mathcal{Y}, h)$ is specified by a fixed state $x \in \mathcal{M}^\tau$. ⋄

[1] We use capital letters to indicate states with a matrix representation.

Remark 2.14: The popular concept of a *framework* [8], [9], [14] is defined by an agent network $\mathcal{N} := (\mathcal{M}^\tau, \mathcal{Y}, h)$ along with a configuration $x \in \mathcal{M}^\tau$. We do not use the notion of a framework in this paper. ⋄

Definition 2.15: For a given agent network $\mathcal{N} := (\mathcal{M}^\tau, \mathcal{Y}, h)$, two configurations $x, x' \in \mathcal{M}^\tau$ are *equivalent* [8] or *indistinguishable* if $y = h(x) = h(x') = y'$. ⋄

A major advantage of our framework is that it accommodates somewhat more general structures than those allowed in existing rigidity formulations; for example, we can consider state-dependent network topologies (Remark 2.16) and interchangeable agents (Remark 2.17).

Remark 2.16: In practical scenarios it is commonly the case that a sensor (such as an onboard camera) has a limited field of view, i.e. that the availability of a measurement y_k depends upon the states of certain agents. Such situations can be appropriately modelled in our framework by augmenting a measurement space $\tilde{\mathcal{Y}}_k$ with an additional point $\aleph$ (i.e. to obtain $\mathcal{Y}_k := \tilde{\mathcal{Y}}_k \cup \{\aleph\}$). This point is used to indicate that the measurement is not available, which can itself be useful knowledge of the system state. ⋄

Remark 2.17: Consider the case of interchangeable agents as described in Example 2.2, where $\sim$ denotes the equivalence relation of agent permutations. Let $\tilde{h} : \mathcal{M}^\wp \to \mathcal{Y}^\tau$ be a continuous sensor modality, and suppose $\tilde{x} \sim \tilde{x}'$ implies that $\tilde{x}, \tilde{x}' \in \mathcal{M}^\wp$ are indistinguishable, i.e. that $\tilde{h}(\tilde{x}) = \tilde{h}(\tilde{x}')$. Let $\pi : \mathcal{M}^\wp \to \mathcal{M}^\tau$ be the natural projection to the quotient space $\mathcal{M}^\wp / \sim$. The universal property of quotient spaces then ensures the existence of a unique continuous map $h : \mathcal{M}^\tau \to \mathcal{Y}^\tau$ such that $\tilde{h} = h \circ \pi$. The existence of such a map h shows that our formulation readily accommodates this scenario. ⋄

Example 2.18: Consider a group of four agents, where agents 1 and 2 are both equipped with a range sensor that does not distinguish between other agents, as in Example 2.11. The quotient structure of the measurement space $\mathcal{Y}_k^\tau$ means that agents 3 and 4 are indistinguishable, and can therefore be regarded as interchangeable. Agents 1 and 2 are *not* indistinguishable because the measurements y_1 and y_2 correspond to measurements from agents 1 and 2, respectively (i.e. the full measurement space has the product topology $\mathcal{Y}^\wp := \mathcal{Y}_1^\tau \times \mathcal{Y}_2^\tau$). However, we could choose to make them indistinguishable by defining an equivalence relation $(y_1, y_2) \sim_y (y_2, y_1)$ on $\mathcal{Y}^\wp$ (independent of the equivalence relation on each $\mathcal{Y}_k^\tau$ described in Example 2.11), and using the measurement space $\mathcal{Y}^\tau := \mathcal{Y}^\wp / \sim_y$. This would enable us to regard agents 1 and 2 as an interchangeable pair, and agents 3 and 4 as another interchangeable pair, with the output map h remaining well-defined on the corresponding quotient space $\mathcal{M}^\tau := \mathcal{M}^\wp / \sim$. In this way, we can create a network of several different types of agents, with all agents of each type being interchangeable. ⋄

The following definition formalises the concept of an agent formation, which is associated with a fixed output $y^\star \in \mathcal{Y}$.

Definition 2.19: For a given agent network $\mathcal{N} := (\mathcal{M}^\tau, \mathcal{Y}, h)$ on a Hausdorff topological space $\mathcal{M}^\tau$, a *formation* $\mathcal{F}(y^\star)$ is defined as the set of configurations $x \in \mathcal{M}^\tau$

such that $h(x) = y^\star$. That is,

$$\mathcal{F}(y^\star) := \{x \in \mathcal{M}^\tau \mid y = h(x) = y^\star\}. \qquad \diamond$$

Conceptually, a configuration fixes the state $x \in \mathcal{M}^\tau$ of the system while a formation is the set of configurations in the pre-image of a fixed output $y^\star \in \mathcal{Y}$. We may alternatively regard $\mathcal{F}(y^\star)$ as the set of configurations that are equivalent to some reference configuration $\mathring{x} \in \mathcal{M}^\tau$ that generates the reference measurement $y^\star = h(\mathring{x})$. It should be noted that a particular specification $y^\star$ may not be realisable; in this case the corresponding formation is the null set.

B. Equivariance and Congruence

We define rigidity with respect to a symmetry of the system. Let $\mathbf{G}$ be a Hausdorff topological group[2] with a continuous group action $\Phi : \mathbf{G} \times \mathcal{M}^\tau \to \mathcal{M}^\tau$. In this paper we shall work with left group actions, but the choice is arbitrary.

Remark 2.20: Although it is quite common to have an individual group action for each agent (i.e. to have Φ composed of n group actions $\phi_i : \mathbf{G} \times \mathcal{M}_i^\tau \to \mathcal{M}_i^\tau$), the more general definition is crucial for enabling other possibilities. For example, one may wish the symmetry to allow permutations of the agent positions, or reflections of an agent's location through a line between two others. $\diamond$

There are two types of invariance that are of interest to our study of rigidity, depending upon the scenario at hand. The first of these is as follows.

Definition 2.21: An output map $h(x)$ is termed *invariant* with respect to a continuous group action Φ of a Hausdorff topological group $\mathbf{G}$ if $h(x) = h(\Phi(S, x))$ for all $S \in \mathbf{G}$ and $x \in \mathcal{M}^\tau$. $\diamond$

Example 2.22: For a range measurement (Example 2.8), we can act on agent states $x_i \in \mathbb{R}^3$ by a rigid-body transformation and (optionally) a reflection,

$$\Phi((Q, \xi), x) := (Qx_1 + \xi, \ldots, Qx_n + \xi).$$

That is, the same element $S = (Q, \xi) \in \mathrm{E}(3)$ (where $Q \in \mathrm{O}(3)$) is applied to each individual state x_i. Then

$$h_k(\Phi(S, x)) = \|Qx_i + \xi - Qx_j - \xi\| = \|x_i - x_j\| = h_k(x),$$

and we confirm the well-known result that distances are invariant to rigid-body transformations and reflections. $\diamond$

Example 2.23: For a relative position measurement between agents in $\mathrm{SE}(3)$ (Example 2.9), the group action associated with a rigid-body transformation is left multiplication by the Lie-group $\mathrm{SE}(3)$, i.e.

$$\Phi(S, X) := (SX_1, \ldots, SX_n)$$

where $S \in \mathrm{SE}(3)$. The relative position measurement is

$$\overline{h_k(\Phi(S, X))} = (X_i^{-1} S^{-1} S X_j)\mathring{\bar{y}}_k = (X_i^{-1} X_j)\mathring{\bar{y}}_k = \bar{y}_k. \quad \diamond$$

In practice, full invariance of the output map may be stronger than required for rigidity of a specific formation. It can be sufficient to have an invariance property hold at the specific output value that defines a formation. This motivates the following definition.

Definition 2.24: An output value $y^\star$ is termed *invariant* with respect to a continuous group action Φ of a Hausdorff topological group $\mathbf{G}$ if, for all $x \in \mathcal{M}^\tau$ such that $h(x) = y^\star$, it holds that $h(\Phi(S, x)) = y^\star$ for all $S \in \mathbf{G}$. $\diamond$

Clearly, every output value is invariant if the associated output map is invariant. The distinction between Definitions 2.21 and 2.24 is illustrated by the following example.

Example 2.25: Let $x_i, x_j \in \mathbb{R}^3$ be the states of two agents with an inertial direction measurement as in Example 2.10. Considering invariance with respect to a Euclidean transform $S = (Q, \xi) \in \mathrm{E}(3)$ (with $\phi_i(S, x_i) := Qx_i + \xi$), we have

$$h_k(\Phi(S, x)) = \frac{Q(x_i - x_j)}{\|x_i - x_j\|} = Qh_k(x) \tag{5}$$

if $x_i \neq x_j$, and $h_k(\Phi(S, x)) = h_k(x) = \aleph$ otherwise. Hence, the sensor modality h_k is invariant with respect to the translational component of $\mathrm{E}(3)$, but not to the rotations or reflections[3]. However, the particular measurement $y_k^\star = \aleph$ *is* invariant with respect to the group action of $\mathrm{E}(3)$. Furthermore, we see from (5) that a fixed inertial direction measurement $y_k^\star \in \mathbb{S}^2$ is invariant with respect to rotations about the axis of that measurement. $\diamond$

Remark 2.26: Consider a Hausdorff topological group $\mathbf{G}$ and a continuous group action $\bar{\Phi} : \mathbf{G} \times \mathcal{M}^\wp \to \mathcal{M}^\wp$. Let $\mathcal{M}^\tau := \mathcal{M}^\wp / \sim$ be a Hausdorff quotient space with an equivalence relation $\sim$. The continuity of the natural projection $\pi : \mathcal{M}^\wp \to \mathcal{M}^\tau$ ensures that the map $\tilde{\Phi} := \pi \circ \bar{\Phi}$ is also continuous. Furthermore, analogously to the case in Remark 2.17, suppose that $\tilde{x} \sim \tilde{x}'$ (where $\tilde{x}, \tilde{x}' \in \mathcal{M}^\wp$) implies $\tilde{\Phi}(S, \tilde{x}) = \tilde{\Phi}(S, \tilde{x}')$ for all $S \in \mathbf{G}$. In this case there exists a unique continuous map $\Phi : \mathbf{G} \times \mathcal{M}^\tau \to \mathcal{M}^\tau$ such that $\Phi(S, \pi(\tilde{x})) := \tilde{\Phi}(S, \tilde{x})$. $\diamond$

Remark 2.27: Consider a Hausdorff topological group $\mathbf{G}$ and a continuous group action $\tilde{\Phi} : \mathbf{G} \times \tilde{\mathcal{M}}^\tau \to \tilde{\mathcal{M}}^\tau$. Suppose we wish to consider the state-space $\mathcal{M}^\tau := \tilde{\mathcal{M}}^\tau \setminus \mathcal{W}$, where $\mathcal{W} \subset \tilde{\mathcal{M}}^\tau$ is an exceptional set. We can use the induced group action $\Phi : \mathbf{G} \times \mathcal{M}^\tau \to \mathcal{M}^\tau, \Phi(S, x) \mapsto \tilde{\Phi}(S, x)$ if $\tilde{\Phi}(S, x) \in \mathcal{M}^\tau$ for all $S \in \mathbf{G}$ and all $x \in \mathcal{M}^\tau$, i.e. if the orbits of $\mathbf{G}$ intersecting $\mathcal{W}$ are contained in $\mathcal{W}$. In particular, note that this condition is always satisfied in the common case where the group action acts on each agent state $x_i \in \mathcal{M}_i^\tau$ in the same way and $\mathcal{W}$ is the set of points where two agent states coincide (as in Example 2.3). $\diamond$

Rigidity of a formation is a property concerned with how the symmetries of the system relate to the formation. The two notions of invariance defined above (Definitions 2.21 and 2.24) lead to two classes of symmetry.

Definition 2.28: Let $\mathcal{N} := (\mathcal{M}^\tau, \mathcal{Y}, h)$ be an agent network on a Hausdorff topological space $\mathcal{M}^\tau$, let $\mathcal{F}(y^\star)$ be a formation of the agent network $\mathcal{N}$, and let $\mathbf{G}$ be a Hausdorff topological group with a continuous group action $\Phi : \mathbf{G} \times \mathcal{M}^\tau \to \mathcal{M}^\tau$.

[2]A topological group is Hausdorff [11, Proposition 12.1.6] and Tychonoff [11, Theorem 12.1.7] if and only if it is a Kolmogorov (T_0) space.

[3]Note that inertial direction measurements are also invariant to scaling, but we have omitted demonstrating this in the interest of space.

(i) If the output map $h(x)$ is invariant with respect to Φ, then we say that the *agent network* $\mathcal{N}$ is *equivariant* with respect to Φ.

(ii) If an output value $y^\star$ is invariant with respect to Φ, then we say that the *formation* $\mathcal{F}(y^\star)$ is *equivariant* with respect to Φ. ⋄

Note that if an agent network is equivariant, then any formation of that agent network is equivariant. We emphasise that equivariance of a formation requires *all* measurements $y_k^\star$ to be invariant to the *same* group action.

The final concept we require to define rigidity is a generalised form of congruence [8].

Definition 2.29: Two configurations $x, x' \in \mathcal{M}^\tau$ of an agent network are *congruent* with respect to a continuous group action $\Phi : \mathbf{G} \times \mathcal{M}^\tau \to \mathcal{M}^\tau$ of a Hausdorff topological group $\mathbf{G}$ if there exists a transform $S \in \mathbf{G}$ such that $\Phi(S, x) = x'$. ⋄

III. Defining rigidity

In this section we provide a generalised definition of rigidity for formations, using the concepts of equivalent and congruent configurations. We then apply group theory to study the relationship between the group symmetry and the space of configurations in a rigid formation.

Definition 3.1: (**Generalised rigidity**) Let $\mathcal{F}(y^\star)$ be a formation that is equivariant with respect to a continuous group action $\Phi : \mathbf{G} \times \mathcal{M}^\tau \to \mathcal{M}^\tau$ of a Hausdorff topological group $\mathbf{G}$. The formation $\mathcal{F}(y^\star)$ is *locally rigid* with respect to Φ if, for any configuration $x \in \mathcal{F}(y^\star)$, there exists an open neighbourhood $\mathcal{U}_x \subseteq \mathcal{M}^\tau$ of x such that all configurations $x' \in (\mathcal{F}(y^\star) \cap \mathcal{U}_x)$ are congruent. That is, there exists $S \in \mathbf{G}$ such that $\Phi(S, x) = x'$. If this holds with $\mathcal{U}_x = \mathcal{M}^\tau$, we say that $\mathcal{F}(y^\star)$ is *globally rigid.* ⋄

We emphasise that global rigidity is a special case of local rigidity. The essence of global rigidity for a formation is that all possible configurations equivalent to an element of the formation are reachable in state-space by applying a suitable symmetry transformation. That is, the group action is transitive on the set of valid configurations, and therefore the formation has the structure of a homogeneous space. We shall explore this insight further after a few examples.

Example 3.2: Consider four agents forming a rectangle in $\mathbb{R}^2$ with nonzero distance constraints (Example 2.8) specified between all pairs (i.e. $m = 6$). This formation $\mathcal{F}(y^\star)$ will be globally rigid with respect to the group action of $\mathrm{E}(2)$. However, it is only locally rigid with respect to the action of $\mathrm{SE}(2)$, because in this case the neighbourhood $\mathcal{U}_x$ about any configuration $x \in \mathcal{F}(y^\star)$ must be small enough not to include a reflected configuration. ⋄

Example 3.3: To demonstrate the flexibility of our framework, we extend the scenario of the previous example by adding a fifth agent with state $X_5 = (R_5, \xi_5) \in \mathrm{SE}(3)$, where the third element of the position $\xi_5 \in \mathbb{R}^3$ is constrained to be positive (i.e. $e_3^\top \xi_5 > 0$, where $e_3 := (0,0,1)^\top$). This scenario might correspond in practice to the case of four ground vehicles (i.e. in the x-y plane) maintaining formation with a single aerial vehicle. Suppose that the aerial vehicle is equipped with a downwards pointing camera that obtains a bearing measurement (in the body-fixed frame) to each of the ground vehicles, modelled as follows:

$$h_k(x_i, X_5) := \frac{R_5^\top(\tilde{x}_i - \xi_5)}{\|\tilde{x}_i - \xi_5\|} \in \mathbb{S}^2.$$

Here, $i \in \{1,2,3,4\}$ and $\tilde{x}_i := (x_i^\top, 0)^\top$ is the embedding of the $\mathbb{R}^2$ state into $\mathbb{R}^3$. Note that the constraint $e_3^\top \xi_5 > 0$ ensures the agents are not co-located. Since the ground vehicles are constrained to the plane, we consider rigidity with respect to $\mathrm{SE}(2)$. The group action on X_5 with $S = (R_S, \xi_S) \in \mathrm{SE}(2)$ is simply given by $\phi_5(S, X_5) := \tilde{S} X_5$, where $\tilde{S}$ denotes the block matrix

$$\tilde{S} := \begin{pmatrix} R_S & 0 & \xi_S \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SE}(3).$$

Using this construction, we determine that the formation is *globally* rigid with respect to the group action of $\mathrm{SE}(2)$; in particular, note that a reflection in $\mathrm{E}(2)$ would result in a state for vehicle 5 that does not lie in $\mathrm{SE}(3)$. This stronger rigidity property is a result of the orientation constraints imposed by the bearing measurements, and cannot be achieved with distance constraints alone. ⋄

Example 3.4: To illustrate the interest in invariant *output values* (rather than invariant *output maps*), consider the case of two vehicles in $\mathbb{R}^2$ (in the x-y plane) with a distance measurement between them. Suppose a third vehicle in $\mathbb{R}^3$ measures the relative height (i.e. the z-component of the relative position) and the inertial direction (Example 2.10) of the first vehicle, with the goal of being positioned directly above it. Although the directional sensor modality is not invariant to rotations, the particular desired measurement $y_k^\star = (0,0,-1)^\top$ is invariant to rotations about the z-axis. Hence, the formation will still be globally rigid with respect to the group $\mathrm{E}(2)$ acting in the x and y directions. ⋄

We define the *stabiliser* of a point $x \in \mathcal{M}^\tau$ as $\operatorname{stab}\Phi_x := \{S \in \mathbf{G} \mid \Phi(S,x) = x\} \subseteq \mathbf{G}$, i.e. the set of transformations in $\mathbf{G}$ that leave x unchanged by the group action. It is well-known (and straightforward to verify) that this is a subgroup of $\mathbf{G}$. Since it is the pre-image of a singleton set in $\mathcal{M}^\tau$ (which is T_1, and indeed Hausdorff), the continuity of $\Phi_x(S) := \Phi(S,x)$ implies that it is closed. It follows that the quotient space $\mathbf{G}/\operatorname{stab}\Phi_x$ is Hausdorff [11, Proposition 7.1.6, note also Proposition 12.3.1]. The following theorem considers some well-known characteristics of homogeneous spaces in the context of globally rigid formations.

Theorem 3.5: Let $\mathcal{F}(y^\star)$ be a formation and let $\mathbf{G}$ be a Hausdorff topological group with a continuous group action $\Phi : \mathbf{G} \times \mathcal{M}^\tau \to \mathcal{M}^\tau$. Then, the following hold:

(i) if $\mathcal{F}(y^\star)$ is globally rigid, then all stabilisers $\operatorname{stab}\Phi_{\mathring{x}}$ with $\mathring{x} \in \mathcal{F}(y^\star)$ are homeomorphic. More specifically, for all $\mathring{x}, \mathring{x}' \in \mathcal{F}(y^\star)$, there exists $S \in \mathbf{G}$ such that $\operatorname{stab}\Phi_{\mathring{x}'} = S \cdot \operatorname{stab}\Phi_{\mathring{x}} \cdot S^{-1}$.

(ii) the formation $\mathcal{F}(y^\star)$ is globally rigid with respect to Φ if and only if there exists some reference state $\mathring{x} \in$

$\mathcal{F}(y^\star)$ such that the mapping

$$\Psi_{\mathring{x}} : \mathbf{G}/\operatorname{stab}\Phi_{\mathring{x}} \to \mathcal{M}^\tau, \Psi_{\mathring{x}}(S \cdot \operatorname{stab}\Phi_{\mathring{x}}) \mapsto \Phi_{\mathring{x}}(S)$$

is continuous and bijective onto $\mathcal{F}(y^\star)$, where $S \in \mathbf{G}$ (see Figure 1).

Proof: To show (i), we note that there exists $S \in \mathbf{G}$ such that $\Phi(S, \mathring{x}) = \mathring{x}'$ since $\mathcal{F}(y^\star)$ is globally rigid. For any $S' \in \operatorname{stab}\Phi_{\mathring{x}'}$, we have $\Phi(S', \mathring{x}') = \mathring{x}'$ and therefore $\Phi(S^{-1} \cdot S' \cdot S, \mathring{x}) = \Phi(S^{-1}, \mathring{x}') = \Phi(S^{-1} \cdot S, \mathring{x}) = \mathring{x}$. This implies that $\operatorname{stab}\Phi_{\mathring{x}'} \subseteq S \cdot \operatorname{stab}\Phi_{\mathring{x}} \cdot S^{-1}$. The analogous argument with $\Phi(S^{-1}, \mathring{x}') = \mathring{x}$ shows that the relation can be reversed, so the two sets are equal. Note that the group operation $(\cdot)$ is a homeomorphism in either of its arguments.

For (ii), the forward implication is well-known [11, p. 352] in the context of orbit spaces. For the reverse implication, we observe that since $\Psi_{\mathring{x}}$ is bijective, then for any $x, x' \in \mathcal{F}(y^\star)$ there exist $S \cdot \operatorname{stab}\Phi_{\mathring{x}}, S' \cdot \operatorname{stab}\Phi_{\mathring{x}} \in (\mathbf{G}/\operatorname{stab}\Phi_{\mathring{x}})$ such that $\Psi_{\mathring{x}}(S \cdot \operatorname{stab}\Phi_{\mathring{x}}) = \Phi_{\mathring{x}}(S) = x$ and $\Psi_{\mathring{x}}(S' \cdot \operatorname{stab}\Phi_{\mathring{x}}) = \Phi_{\mathring{x}}(S') = x'$. Thus, we have $\Phi(S' \cdot S^{-1}, x) = \Phi(S', \mathring{x}) = x'$. The formation $\mathcal{F}(y^\star)$ is equivariant since it is the image of $\Psi_{\mathring{x}}$ (Figure 1). It follows that $\mathcal{F}(y^\star)$ is globally rigid. ■

Remark 3.6: In Figure 1, the canonical projection π is continuous and open [11, Proposition 12.3.1]. Therefore, if $\mathcal{F}(y^\star)$ is globally rigid then $\Psi_{\mathring{x}}$ will be a homeomorphism if and only if $\Phi_{\mathring{x}}$ is also open. ⋄

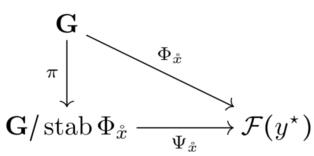

Fig. 1. Mappings between spaces, with a globally rigid formation $\mathcal{F}(y^\star)$. Here, π is the canonical projection, and the diagram commutes.

Example 3.7: The idea of Theorem 3.5 can be illustrated by returning to Example 3.2. For the given situation, the stabiliser of any point $\mathring{x} \in \mathcal{F}(y^\star)$ is the identity of $\mathrm{E}(2)$. However, if the agent states are elements of $\mathbb{R}^3$, the formation is then globally rigid with respect to $\mathrm{E}(3)$. In this case, the stabiliser of $\mathring{x} \in \mathcal{F}(y^\star)$ would include any reflection combined with the $\mathrm{SE}(3)$ action that compensates for the change induced by that reflection. The existence of such an $\mathrm{SE}(3)$ transform is a consequence of the formation being planar; if the fourth vehicle lay outside the plane of the other three, no $\mathrm{SE}(3)$ transform following a reflection would be able to return the vehicle to the original "side" of that plane (note that the plane is fixed to the other three vehicles, not fixed to the inertial frame). ⋄

IV. Path-rigidity

In this section we introduce the concept of *path-rigidity*, which is a stronger property than global rigidity. In the classical literature, this concept is overlooked in favour of *infinitesimal rigidity* [8], but it is of significant interest in the general case where h may not be differentiable. Path-rigidity is particularly relevant to trajectory planning tasks because it guarantees that the agents can continuously transition between any two configurations of the formation without breaking the state constraints. We begin with the formal definitions of continuous congruence and path-rigidity.

Definition 4.1: For a given agent network $\mathcal{N} := (\mathcal{M}^\tau, \mathcal{Y}, h)$ and a Hausdorff topological group $\mathbf{G}$ with continuous group action $\Phi : \mathbf{G} \times \mathcal{M}^\tau \to \mathcal{M}^\tau$, two configurations $x, x' \in \mathcal{M}^\tau$ of $\mathcal{N}$ are *continuously congruent* with respect to Φ if there exists a continuous parametrised function $\sigma(t) : [0,1] \to \mathbf{G}$ such that $\sigma(0) = \iota$ (where $\iota \in \mathbf{G}$ denotes the identity) and $\Phi(\sigma(1), x) = x'$. ⋄

Definition 4.2: (**Path-rigidity**) Let $\mathcal{F}(y^\star)$ be a formation that is equivariant with respect to a continuous group action $\Phi : \mathbf{G} \times \mathcal{M}^\tau \to \mathcal{M}^\tau$ of a Hausdorff topological group $\mathbf{G}$. The formation $\mathcal{F}(y^\star)$ is *path-rigid* with respect to Φ if all configurations $x, x' \in \mathcal{F}(y^\star)$ are continuously congruent. ⋄

Clearly, path-rigidity implies global rigidity. A simple topological characterisation of path-rigidity is as follows.

Lemma 4.3: Let $\mathbf{G}$ be a Hausdorff topological group with a continuous group action $\Phi : \mathbf{G} \times \mathcal{M}^\tau \to \mathcal{M}^\tau$. Then, a globally rigid formation $\mathcal{F}(y^\star)$ is path-rigid with respect to Φ if and only if, for any configuration $\mathring{x} \in \mathcal{F}(y^\star)$, the quotient $\mathbf{G}/\operatorname{stab}\Phi_{\mathring{x}}$ is path-connected.

Proof: The forward implication follows by projecting the path in $\mathbf{G}$ onto $\mathbf{G}/\operatorname{stab}\Phi_{\mathring{x}}$ (cf. Figure 1). For the reverse implication, recall from Theorem 3.5 that the stabilisers of each $\mathring{x} \in \mathcal{F}(y^\star)$ intersect the same number of path-connected components (since the group operation $(\cdot)$ is a homeomorphism), and that $\Psi_{\mathring{x}} : \mathbf{G}/\operatorname{stab}\Phi_{\mathring{x}} \to \mathcal{F}(y^\star)$ is a continuous bijection. Since the quotient $\mathbf{G}/\operatorname{stab}\Phi_{\mathring{x}}$ is path-connected, for any two points $x, x' \in \mathcal{F}(y^\star)$ there exists a path $\sigma : [0,1] \to \mathbf{G}/\operatorname{stab}\Phi_{\mathring{x}}$ such that $\Psi_{\mathring{x}} \circ \sigma$ is a path from x to x'. By the universal property of quotients, σ lifts to a continuous path $\tilde{\sigma} : [0,1] \to \mathbf{G}$, which will have $\tilde{\sigma}(0) = \iota \in \mathbf{G}$ and $\Phi(\tilde{\sigma}(1), x) = x'$. ■

A. Group theoretic analysis

In this subsection we prove a useful group-theoretic result (see Theorem 4.7) that provides an algebraic criterion for the application of Lemma 4.3.

For any group, we can define $\mathbf{G}^0$ as the connected component [11] of the identity, which is closed since connected components are closed [11, Proposition 3.2.2]. Similarly, we denote $\mathbf{G}^1$ as the path-connected component [11] of the identity. Note that $\mathbf{G}^1 \subseteq \mathbf{G}^0$. It is well-known that both $\mathbf{G}^0$ and $\mathbf{G}^1$ are normal subgroups of $\mathbf{G}$ [11, Proposition 12.2.4, Exercise 21 from §12.2].

Our analysis concerns the quotient space $\mathbf{G}/\mathbf{H}$ for a closed subgroup $\mathbf{H} \subseteq \mathbf{G}$. It is well-known that $\mathbf{H}$ being closed is necessary and sufficient for this quotient to be Hausdorff [11, Proposition 7.1.6 with Proposition 12.3.1]. As a simplifying assumption, we suppose that $\mathbf{G}/\mathbf{H}$ is path-connected if it is connected. Proposition 4.4 presents a useful sufficient condition for this to hold.

Proposition 4.4: Let $\mathbf{G}$ be a Hausdorff topological group with a closed subgroup $\mathbf{H} \subseteq \mathbf{G}$, and suppose that the quotient $\mathbf{G}/\mathbf{H}$ is connected. Then, $\mathbf{G}/\mathbf{H}$ is path-connected if the path-connected component $\mathbf{G}^1$ of the identity is open in $\mathbf{G}$.

Proof: The coset $G \cdot \mathbf{G}^1$ is an open, path-connected component of $\mathbf{G}$ since left multiplication by G is a homeomorphism. Since $G \in G \cdot \mathbf{G}^1$, every path-connected component of $\mathbf{G}$ is of this form. In particular, they are all open, and as a consequence they are all closed. It follows that all connected components of $\mathbf{G}$ are path-connected.

Note that the canonical projection $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is a continuous and open surjection [11, Proposition 12.3.1]. It follows from the continuity of π that the image of a path-connected component is path-connected [11, Proposition 3.3.5]. Also note that if two (or more) path-connected subspaces (in $\mathbf{G}/\mathbf{H}$) share a point, their union is path-connected [11, Corollary 3.3.3].

Now suppose, for a contradiction, that $\mathbf{G}/\mathbf{H}$ is not path-connected. There must then exist two complementary collections $\{\mathbf{G}_{a\in\mathcal{I}}\}$ and $\{\mathbf{G}_{b\in\mathcal{I}}\}$ of (path-)connected components of $\mathbf{G}$ whose images (under π) are disjoint (here, $\mathcal{I}$ denotes an index set). Since π is open these images are open, and since π is a surjection these images form a partition of $\mathbf{G}/\mathbf{H}$. This would imply that $\mathbf{G}/\mathbf{H}$ is not connected, which contradicts the assumption of the proposition. ∎

Note that in the particularly common case where $\mathbf{G}$ is locally path-connected, $\mathbf{G}^1$ is open [11, Corollary 3.4.7]. In the sequel, it is convenient to assume that $\mathbf{G}^1$ is open rather than the more general requirement that $\mathbf{G}/\mathbf{H}$ be path-connected if it is connected. The authors are not aware of any practical scenario where this distinction is relevant, but all following results in this paper hold for the broader case.

The following proposition concerns the relationship between the connected components of a group $\mathbf{G}$ and those of a closed subgroup $\mathbf{H}$.

Proposition 4.5: Let $\mathbf{G}$ be a Hausdorff topological group and $\mathbf{H} \subseteq \mathbf{G}$ a closed subgroup. Let $\mathbf{G}^0$ and $\mathbf{H}^0$ be the connected components of the identities in $\mathbf{G}$ and $\mathbf{H}$, respectively. Then the subgroup $\mathbf{H}\cap\mathbf{G}^0 \subseteq \mathbf{H}$ is normal in $\mathbf{H}$ and the group homomorphism

$$\alpha : \mathbf{H}/\mathbf{H}^0 \to \mathbf{H}/(\mathbf{H} \cap \mathbf{G}^0), H \cdot \mathbf{H}^0 \mapsto H \cdot (\mathbf{H} \cap \mathbf{G}^0) \quad (6)$$

is well-defined and surjective for $H \in \mathbf{H}$. Furthermore, the group homomorphism

$$\beta : \mathbf{H}/(\mathbf{H} \cap \mathbf{G}^0) \to \mathbf{G}/\mathbf{G}^0, H \cdot (\mathbf{H} \cap \mathbf{G}^0) \mapsto H \cdot \mathbf{G}^0 \quad (7)$$

is well-defined and injective.

Proof: Note that $H \cdot (\mathbf{H}\cap\mathbf{G}^0) \cdot H^{-1} \subseteq H \cdot \mathbf{H} \cdot H^{-1} \subseteq \mathbf{H}$ since $\mathbf{H}$ is a group and $H\cdot(\mathbf{H}\cap\mathbf{G}^0)\cdot H^{-1} \subseteq H\cdot\mathbf{G}^0\cdot H^{-1} \subseteq \mathbf{G}^0$ since $\mathbf{G}^0$ is normal in $\mathbf{G}$. It follows that $H\cdot(\mathbf{H}\cap\mathbf{G}^0)\cdot H^{-1} \subseteq \mathbf{H}\cap\mathbf{G}^0$ and so $\mathbf{H}\cap\mathbf{G}^0$ is normal in $\mathbf{H}$.

Let $H_1, H_2 \in \mathbf{H}$ with $H_1 \cdot \mathbf{H}^0 = H_2 \cdot \mathbf{H}^0$. Then there exists $H_3 \in \mathbf{H}^0$ such that $H_1 = H_2 \cdot H_3$. Since $\mathbf{H}^0$ is path-connected and contains the identity element of $\mathbf{H}$ and $\mathbf{G}$, we have $\mathbf{H}^0 \subseteq \mathbf{G}^0$ and hence $H_3 \in \mathbf{H}\cap\mathbf{G}^0$. Therefore $H_1\cdot(\mathbf{H}\cap\mathbf{G}^0) = H_2\cdot(\mathbf{H}\cap\mathbf{G}^0)$ and so α is well-defined. It is surjective since $\mathbf{H}^0 \subseteq \mathbf{H}\cap\mathbf{G}^0$.

Now consider $H_1, H_2 \in \mathbf{H}$ with $H_1\cdot(\mathbf{H}\cap\mathbf{G}^0) = H_2\cdot(\mathbf{H}\cap\mathbf{G}^0)$. Then there exists $H_3 \in \mathbf{H}\cap\mathbf{G}^0$ with $H_1 = H_2\cdot H_3$. Since $H_3 \in \mathbf{G}^0$, this implies that $\beta(H_1 \cdot (\mathbf{H}\cap\mathbf{G}^0)) = H_1 \cdot \mathbf{G}^0 = H_2 \cdot \mathbf{G}^0 = \beta(H_2 \cdot (\mathbf{H}\cap\mathbf{G}^0))$ and so β is well-defined.

Finally, let $H_1, H_2 \in \mathbf{H}$ with $\beta(H_1\cdot(\mathbf{H}\cap\mathbf{G}^0)) = H_1\cdot\mathbf{G}^0 = H_2\cdot\mathbf{G}^0 = \beta(H_2\cdot(\mathbf{H}\cap\mathbf{G}^0))$. Then there exists $H_3 \in \mathbf{G}^0$ such that $H_1 = H_2 \cdot H_3$. This implies that $H_3 = H_2^{-1} \cdot H_1 \in \mathbf{H}$ and hence that $H_3 \in \mathbf{H}\cap\mathbf{G}^0$. It follows that $H_1 \cdot (\mathbf{H}\cap\mathbf{G}^0) = H_2 \cdot (\mathbf{H}\cap\mathbf{G}^0)$ and therefore β is injective. ∎

Definition 4.6: Let $\mathbf{G}$ be a Hausdorff topological group and let $\mathbf{G}^0$ be the connected component of the identity. The *component group* of $\mathbf{G}$ is defined as $\pi_0(\mathbf{G}) \coloneqq \mathbf{G}/\mathbf{G}^0$. ⋄

For a subgroup $\mathbf{H} \subseteq \mathbf{G}$ we define the following homomorphism on the component groups:

$$\pi_0^{\mathrm{id}} : \pi_0(\mathbf{H}) \to \pi_0(\mathbf{G}), H \cdot \mathbf{H}^0 \mapsto \mathrm{id}(H) \cdot \mathbf{G}^0. \quad (8)$$

Here, $\mathrm{id} : \mathbf{H} \hookrightarrow \mathbf{G}, H \mapsto H$ is the inclusion group homomorphism. We are now ready to present the main result.

Theorem 4.7: Let $\mathbf{G}$ be a Hausdorff topological group, let $\mathbf{H}$ be a closed subgroup of $\mathbf{G}$, and let $\mathbf{G}^0$ and $\mathbf{H}^0$ be the connected components of the identities in $\mathbf{G}$ and $\mathbf{H}$, respectively. Assume the path-connected component $\mathbf{G}^1$ of the identity in $\mathbf{G}$ is open. Then the following are equivalent:

(i) The homogeneous space $\mathbf{G}/\mathbf{H}$ is connected.
(ii) The subgroup $\mathbf{H}$ contains an element from every connected component of $\mathbf{G}$.
(iii) The homomorphism π_0^{id} (8) of component groups is surjective.
(iv) The homomorphism β (7) is an isomorphism.
(v) For every $G \in \mathbf{G}$ there exist $H \in \mathbf{H}$ and $G^0 \in \mathbf{G}^0$ such that $G = H \cdot G^0$.

Proof: The proof is given as a sequence of implications and equivalences.

(i) ⇒ (ii): By Proposition 4.4, $\mathbf{G}/\mathbf{H}$ is path-connected. Let $\sigma : [0,1] \to \mathbf{G}/\mathbf{H}$ be a continuous path connecting $\iota\cdot\mathbf{H}$ to any $G\cdot\mathbf{H}$ in $\mathbf{G}/\mathbf{H}$. By the universal property of quotients, the path σ lifts to a continuous path $\tilde{\sigma} : [0,1] \to \mathbf{G}$ such that $\sigma = \pi\circ\tilde{\sigma}$, where $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is the canonical projection. By design, the path $\tilde{\sigma}$ connects some element $\iota \cdot H_1 = H_1 \in \mathbf{H} \subseteq \mathbf{G}$ to some element $G \cdot H_2 \in \mathbf{G}$, where $H_2 \in \mathbf{H}$. It follows that $\tilde{\sigma} \cdot H_2^{-1}$ is a continuous path connecting $H_1 \cdot H_2^{-1} \in \mathbf{H} \subseteq \mathbf{G}$ to $G \in \mathbf{G}_1 \subseteq \mathbf{G}$, where $\mathbf{G}_1$ is the path-connected component of G. This implies that $H_1 \cdot H_2^{-1} \in \mathbf{G}_1 \cap \mathbf{H}$. Since G is arbitrary, every path-connected component $\mathbf{G}_1$ contains an element of $\mathbf{H}$.

(ii) ⇒ (i): Let $\mathbf{G}_1$, $\mathbf{G}_2$ be two path-connected components of $\mathbf{G}$ (we allow $\mathbf{G}_1 = \mathbf{G}_2$), with $G_1 \in \mathbf{G}_1$ and $G_2 \in \mathbf{G}_2$. Pick $H_1 \in \mathbf{G}_1 \cap \mathbf{H}$ and $H_2 \in \mathbf{G}_2 \cap \mathbf{H}$. The map $\gamma : \mathbf{G} \to \mathbf{G}$, $G \mapsto G \cdot (H_1^{-1} \cdot H_2)$ is a homeomorphism, so the image of $\mathbf{G}_1$ is a path-connected component of $\mathbf{G}$. Since $H_2 = H_1\cdot(H_1^{-1}\cdot H_2) = \gamma(H_1)$, we have $\gamma(\mathbf{G}_1) = \mathbf{G}_2$. In particular, $\gamma(G_1) = G_1\cdot(H_1^{-1}\cdot H_2) \in \mathbf{G}_2$ and hence there is a continuous path $\tilde{\sigma} : [0,1] \to \mathbf{G}$ connecting $G_1 \cdot (H_1^{-1} \cdot H_2)$ to G_2. Let $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H}$ be the canonical projection, and then $\sigma \coloneqq \pi\circ\tilde{\sigma}$ is a continuous path connecting $G_1 \cdot \mathbf{H}$ to $G_2 \cdot \mathbf{H}$ in $\mathbf{G}/\mathbf{H}$. Since G_1 and G_2 are arbitrary, the quotient $\mathbf{G}/\mathbf{H}$ is path-connected.

(ii) ⇒ (iii): For any $G \in \mathbf{G}$ there exists $H \in \mathbf{H}$ with $H \in G\cdot\mathbf{G}^0$. This implies that $\pi_0^{\mathrm{id}}(H\cdot\mathbf{H}^0) = H\cdot\mathbf{G}^0 = G\cdot\mathbf{G}^0$, and since $G \in \mathbf{G}$ is arbitrary this means that π_0^{id} is surjective.

(iii) $\Rightarrow$ (ii): For any $G \in \mathbf{G}$ there exists $H \in \mathbf{H}$ such that $G \cdot \mathbf{G}^0 = H \cdot \mathbf{G}^0 = \pi_0^{\mathrm{id}}(H \cdot \mathbf{H}^0)$. In particular, $H \in G \cdot \mathbf{G}^0$. Since $G \in \mathbf{G}$ is arbitrary, for every connected component $G \cdot \mathbf{G}^0$ there exists $H \in \mathbf{H}$ with $H \in G \cdot \mathbf{G}^0$.

(iii) $\Leftrightarrow$ (iv): By definition the homomorphism π_0^{id} factors as $\pi_0^{\mathrm{id}} = \beta \circ \alpha$ with α surjective. Hence, π_0^{id} is surjective if and only if β surjective. Since β is always injective, this is the case if and only if β is an isomorphism.

(iv) $\Rightarrow$ (v): For any $G \in \mathbf{G}$ there exists $H \in \mathbf{H}$ such that $G \cdot \mathbf{G}^0 = \beta(H \cdot (\mathbf{H} \cap \mathbf{G}^0)) = H \cdot \mathbf{G}^0$. Hence, there exists $G^0 \in \mathbf{G}^0$ such that $G = H \cdot G^0$.

(v) $\Rightarrow$ (iv): For any $G \in \mathbf{G}$ there exists $H \in \mathbf{H}$ and $G^0 \in \mathbf{G}^0$ such that $G = H \cdot G^0$. This implies $G \cdot \mathbf{G}^0 = H \cdot \mathbf{G}^0 = \beta(H \cdot (\mathbf{H} \cap \mathbf{G}^0))$. Since $G \in \mathbf{G}$ is arbitrary, β is surjective and hence an isomorphism. ■

B. An algebraic characterisation of path-rigidity

In this subsection we employ Theorem 4.7 with $\mathbf{H} = \operatorname{stab} \Phi_{\mathring{x}}$ to acquire the following algebraic characterisation for path-rigid formations.

Corollary 4.8: Let $\mathbf{G}$ be a Hausdorff topological group with a continuous group action $\Phi : \mathbf{G} \times \mathcal{M}^\tau \to \mathcal{M}^\tau$. Assume the path-connected component $\mathbf{G}^1$ of the identity in $\mathbf{G}$ is open. Then, a globally rigid formation $\mathcal{F}(y^\star)$ is path-rigid with respect to Φ if and only if the stabiliser of any configuration $\mathring{x} \in \mathcal{F}(y^\star)$ contains at least one element from every (path-)connected component of $\mathbf{G}$.

Proof: By Theorem 4.7, the quotient $\mathbf{G}/\operatorname{stab} \Phi_{\mathring{x}}$ is path-connected if and only if $\operatorname{stab} \Phi_{\mathring{x}}$ contains an element from every (path-)connected component of $\mathbf{G}$. The result follows from Lemma 4.3. ■

Due to space constraints, we shall illustrate the application of this result with just a few simple examples.

Example 4.9: In Example 3.3 the formation is globally rigid with respect to $\mathrm{SE}(2)$. This group has only one (path-)connected component and the stabiliser trivially contains the identity, so the formation is path-rigid. ◇

Example 4.10: Consider four agents in $\mathbb{R}^3$ with a distance measurement between each of the six pairs, and suppose we constrain all distances to be equal to 2. This results in a triangular pyramid formation that is globally rigid with respect to the action of $\mathrm{E}(3)$. Since the stabiliser is the identity, the formation is not path-rigid.

Now suppose that we quotient the output space by the equivalence relation $y \sim_y y' \Leftrightarrow \exists \sigma \in \mathbf{P}_6(y) : y' = \sigma(y)$, i.e. let $\mathcal{Y}^\tau := \mathcal{Y}^\wp / \sim_y$, and further suppose that agents 1 and 2 are interchangeable (see Example 2.18). Clearly, the specified formation is still a triangular pyramid that is globally rigid with respect to the group action of $\mathrm{E}(3)$. Now, if we align the pyramid such that $x_1 = (-1,0,0)^\top$ and $x_2 = (1,0,0)^\top$, both x_3 and x_4 will lie in the y-z plane. Reflecting the formation through this plane will therefore only switch the positions of agents 1 and 2, which are interchangeable, and hence this reflection is an element of the stabiliser. From Corollary 4.8 it follows that the formation is path-rigid. ◇

As illustrated by the last example, a nice feature of the corollary is that we only need to consider the stabiliser at a single configuration of the formation. The authors believe the corollary will be of particular appeal for more complex scenarios involving non-product topologies on the state-space.

V. Conclusion

We have presented a generalised formulation for the concept of rigid formations. The definition of rigidity is associated with a Hausdorff topological group and a continuous, transitive group action on the space of valid configurations, to which the constraints specifying the formation are invariant. Our framework allows an extremely broad class of state-spaces and output spaces, requiring only that the former are Hausdorff. The framework therefore enables a very wide range of possible scenarios in fields such as formation control. In addition, we introduced the concept of *path-rigidity*, where agents can move continuously between any two configurations of the formation without breaking it. Our final result provides a very useful characterisation of this property in terms of the stabiliser of the group action.

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