A non-linear internal model principle for observers ⋆

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1. EXTENDED ABSTRACT

State observer design for non-linear systems is concerned with the question of how to construct a dynamical system, the *observer*, that takes as input both the input u and the output y of a given non-linear control system

$$\begin{aligned} \dot{x} &= f(x, u, t), \\ y &= h(x, u, t), \end{aligned} \tag{1}$$

and produces as its output an estimate $\hat{x}$ of the state variable x. The notation in (1) is deliberately generic since many variations of this problem are studied in the literature that differ in terms of the spaces that the variables x, u, and y live in, the assumed properties of the functions f and h, or the notion of what constitutes a good estimate $\hat{x}$ of x.

One way to approach this problem is to make the Ansatz

$$\dot{\hat{x}} = f(\hat{x}, u, t) + \Delta(\hat{x}, u, y, t), \tag{2}$$

where the *internal model term* $f(\hat{x}, u, t)$ simulates the observed system, or *plant*, (1) and the *correction term* $\Delta(\hat{x}, u, y, t)$ is zero along trajectories of (1). In the linear context this idea dates back to Luenberger (1964), and in the non-linear context at least to Kou (1973), see also Thau (1973). In the case where $x, \hat{x} \in \mathbb{R}^n$ and where we are seeking an asymptotic observer, the problem is now to construct a correction term $\Delta(\hat{x}, u, y, t)$ such that the *observer error* $e := \hat{x} - x$ fulfils

$$\lim_{t \to \infty} e(t) = 0 \tag{3}$$

for all (admissible) choices of $x(0)$, $\hat{x}(0)$ and u.

The significance of $\Delta(\hat{x}, u, y, t)$ being zero along trajectories of (1) is that, together with uniqueness of solutions, it implies the *tracking property*

$$\hat{x}(0) = x(0) \implies \hat{x}(t) = x(t) \text{ for all } t. \tag{4}$$

In other words, the behaviour (set of trajectories) of the observer (2) contains the behaviour (set of trajectories) of the plant (1), i.e. an *internal model* of the plant.

An obvious question now is to what extent such an internal model plus correction term design is *necessary*. If we have a general asymptotic observer

$$\dot{\hat{x}} = \hat{f}(\hat{x}, u, y, t) \tag{5}$$

for which (3) holds, does the right hand side $\hat{f}$ *always* split as in (2)? An affirmative answer to such a question is called an *internal model principle for observers*. Note that such a result depends on the classes of plants and observers under consideration as well as on the notion of what constitutes a good estimate.

For linear systems, several such internal model principles were proved in Trumpf et al. (2014), covering the most common notions of good estimate: asymptotic, dead-beat, and exact observers. See also the even more general results in Blumthaler and Trumpf (2014). In the linear case, observers do not necessarily contain *full* internal models of the plant but internal models of significant parts of the plant behaviour. For the details see Trumpf et al. (2014) or Blumthaler and Trumpf (2014).

In this work, we will present a general internal model principle for observers formulated in a purely set theoretic generalisation of behavioural observer theory. We show that the historic focus on the linear case has somewhat obscured what is in essence a surprisingly simple theory, at least once the manifold implications of linearity have been disentangled and only the strictly necessary components kept and generalized. We recover the known linear results as special cases of our general result and also derive a novel internal model principle for non-linear kinematic systems on differentiable manifolds. To our best knowledge, this is the first non-linear internal model principle for observers in the literature.

Our theory proceeds from the observations that (3) defines an equivalence relation on the set of state trajectories, i.e. a notion of which pairs of trajectories are *close* to each other, and that sets of trajectories form a *poset* (partially ordered set) under set inclusion. We define the *saturation* of a given behaviour (set of trajectories) as the set closure under the closeness relation and use this concept to define what we call the *radical set* associated with a given behaviour (set of trajectories). The space of saturations of behaviours is a poset under set inclusion. We show that the space of radical sets also carries a natural poset structure. The case where the poset of saturations and the poset of radical sets are isomorphic via the natural isomorphism is of particular interest. We say that the poset of radical sets admits *local poset sections* in this case. This property holds for linear systems as well as for kinematic systems on differentiable manifolds.

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The general internal model principle for observers now states that if the poset of radical sets admits local poset sections and if the radical sets of behaviours are well-founded (the latter is a standard property in the theory of posets that implies the existence of minimal elements) then any non-intrusive, observable observer behaviour contains a minimal radical of the plant behaviour.

Focusing on the case of asymptotic observer design, for linear systems the minimal radical is unique and equals the anti-stabilizable part of the plant behaviour, cf. (Trumpf et al., 2014, Theorem 5.6). We show that for kinematic systems on differentiable manifolds the minimal radical is again unique and, in this case, equals the plant behaviour. It follows that any asymptotic observer for a non-linear kinematic system on a differentiable manifold contains a full internal model of the plant.

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