An Internal Model Principle for Non-linear Observers for Repeatable Systems on Manifolds

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1. EXTENDED ABSTRACT

Internal model principles provide necessary conditions on the structure of controllers or observers that achieve a certain control or estimation goal. In formulating such principles, care must be taken in clearly defining the properties of the plant under consideration as well as the set of candidate controllers or observers.

1.1 The plant

We consider an arbitrary but fixed *plant* of the form

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= h(x), \end{aligned} \tag{1}$$

where $f\colon M \times U \longrightarrow TM$ is a continuous and uniformly locally Lipschitz family of vector fields on the smooth manifold M, indexed by a topological space U, and $h\colon M \longrightarrow Y$ is a continuous map into a topological space Y. To deal with a subtle issue that will be familiar to readers who have studied Lyapunov theory for time-varying systems, we choose and fix a global minimum time $t_{\min} \in \mathbb{R}$. It is well known that for every Lebesgue measurable and locally essentially bounded input function $u\colon I \longrightarrow U$, $I \subset [t_{\min}, \infty)$ an interval, and for every choice of initial data $(t_0, x_0) \in I \times M$, there exists a unique absolutely continuous maximal solution $x_{(t_0,x_0,u)}$ of system (1) that is defined on an interval $J_{(t_0,x_0,u)} \subset I$ with $t_0 \in J_{(t_0,x_0,u)}$ and fulfils $x_{(t_0,x_0,u)}(t_0) = x_0$. This solution depends continuously on the initial value x_0, see for example Jafarpour and Lewis (2014). A solution $x_{(t_0,x_0,u)}$ gives rise to the associated output function $y_{(t_0,x_0,u)}$. To simplify the notation, we will drop the index (t_0, x_0, u) where there is no room for confusion.

There are well known conditions on the family f of vector fields and/or the manifold M that guarantee existence of solutions for arbitrarily large times, i.e. the absence of finite escape. For example, a linear growth bound on f, or compactness of M. For the purpose of *asymptotic* observer theory, the *behavior* P of system (1) is defined as the set of all trajectories $(u, y_{(t_0,x_0,u)}, x_{(t_0,x_0,u)})$ as above where $J_{(t_0,x_0,u)}$ contains an interval of the form $[t_{\text{start}}, \infty)$. We assume $P \neq \emptyset$. We define the set $R(P) \subset M$ of states *visited* by P as the set of $x_0 \in M$ such that there exist $(u, y, x) \in P$ and $t_0 \in J$ with $x(t_0) = x_0$.

Before we introduce the crucial system property we will require of system (1), we define the *reachability relation* $\to$ on M by $x_1 \to x_2$ if there exist $t_2 > t_1 \in \mathbb{R}$ and a trajectory $(u, y, x) \in P$ with $t_1, t_2 \in J$, $x(t_1) = x_1$ and $x(t_2) = x_2$. Because system (1) is time-invariant, it is easy to see that the relation $\to$ is transitive.

In this paper, we call system (1) *controllable* if for every $x_1, x_2 \in R(P)$ and every $t_1 \geq t_{\min}$ there exist $t_2 > t_1$ and a trajectory $(u, y, x) \in P$ with $t_1, t_2 \in J$, $x(t_1) = x_1$ and $x(t_2) = x_2$. It can be shown that system (1) is controllable if and only if the reachability relation is trivial on $R(P)$, i.e. $x_1 \to x_2$ for all $x_1, x_2 \in R(P)$.

We call system (1) *repeatable* if for any given plant trajectory $(u, y, x) \in P$, all $t_1 \in J$ and all $\Delta_{t,1} > 0$ there exist another plant trajectory $(\tilde{u}, \tilde{y}, \tilde{x}) \in P$ with $t_1 \in \tilde{J}$ and $\Delta_{t,2} > 0$ such that $(\tilde{u}, \tilde{x})|_{[t_1, t_1+\Delta_{t,1})} = (u, x)|_{[t_1, t_1+\Delta_{t,1})}$ and $(\tilde{u}(t_1 + \Delta_{t,1} + \Delta_{t,2}), \tilde{x}(t_1 + \Delta_{t,1} + \Delta_{t,2})) = (u(t_1), x(t_1))$. In words, system (1) is repeatable if we can follow any of its trajectories for some time $\Delta_{t,1}$ and then return to the beginning of the trajectory piece after some additional time $\Delta_{t,2}$, in such a way that we could repeat the trajectory piece once, or even a countable number of times, again. It can be shown that system (1) is repeatable if and only if the reachability relation $\to$ is an equivalence relation on $R(P)$.[1] In particular, it follows that controllable systems are repeatable. It can also be shown that kinematic (input-linear) systems are repeatable whether or not they are controllable. In the following, we assume system (1) to be repeatable.

1.2 Candidate observers

We study candidate observers of the general form

$$\begin{aligned} \dot{z} &= \hat{f}(z,u,y), \\ \hat{x} &= \pi(z), \end{aligned} \tag{2}$$

where $\pi\colon L \longrightarrow M$ is a smooth fiber bundle and $\hat{f}\colon L \times U \times Y \longrightarrow TL$ is a continuous and uniformly locally Lipschitz family of vector fields on L indexed by $U \times Y$. This includes the nonlinear observers for systems with

[1] In his book, Sontag (1998) calls a system *reversible* if the reachability relation is symmetric. In the setting of Sontag's book, $R(P) = M = \mathbb{R}^n$. Reflexivity on $R(P)$ follows from transitivity and symmetry.

symmetry that have been extensively studied by the first author and coauthors, see e.g. Mahony et al. (2013) or Trumpf (2018), but is a much larger class. The fiber bundle condition on π relates to the relatively little known (and only recently rigorously proved) result that a submersion $\pi\colon L \longrightarrow M$ between smooth manifolds allows a complete Ehresmann connection if and only if it is a fiber bundle, see del Hoyo (2016). A complete Ehresmann connection guarantees that integral curves of a lifted vector field on L have maximal existence intervals that are no shorter than the maximal existence interval of the corresponding integral curve on M, making it meaningful to interpret the output trajectory $\hat{x}$ of the observer as an estimate for the plant's state trajectory x. We define the behavior $\hat{O}$ of system (2) as the set of trajectories (u, y, z) whose maximal existence interval contains an interval of the form $[t_{\text{start}}, \infty)$, and the (projected) observer behaviour O as the corresponding set of trajectories $(u, y, \hat{x})$.

To compare plant and observer trajectories, we fix a metric $d\colon M \times M \longrightarrow \mathbb{R}_{\geq 0}$ on M and consider the observer property $\lim_{t\to\infty} d(x(t), \hat{x}(t)) = 0$ for all corresponding trajectories $(u, y, x) \in P$ and $(u, y, \hat{x}) \in O$.

Without any further assumptions on system (2), this is not a very useful property as, for example, the empty set $O = \emptyset$ fulfils this but is arguably not very useful as an observer. Note that $O = \emptyset$ could result even if the system equations (2) themselves are meaningful, for example if all observer trajectories suffer from finite escape. It makes sense to require that for every plant trajectory $(u, y, x) \in P$ there exists at least one corresponding observer trajectory $(u, y, \hat{x}) \in O$.

A further complication arises from manifold topology. It is well known that *global* asymptotic observers with continuous right hand side cannot exist on certain smooth manifolds, see Bhat and Bernstein (2000). To address this, our theory allows for an *exceptional set* $E_0 \subset M \times L$ of excluded initial conditions with the only required property that $d(x, \pi(z)) = 0$ implies $(x, z) \notin E_0$, i.e. we cannot exclude pairs of initial conditions (x_0, z_0) with zero error.

Fixing such an exceptional set E_0 amounts to tweaking the notion of asymptotic stability to be, for example, almost global asymptotic stability or even only local asymptotic stability if we pick a large set E_0. Global asymptotic stability corresponds to the choice $E_0 = \emptyset$ in cases where this makes sense. We define the restricted behavior $\hat{O}(E_0)$ of system (2) as the set of trajectories $(u, y, z) \in \hat{O}$ such that there exists a corresponding trajectory $(u, y, x) \in P$ with $(x(t_{\text{start}}), z(t_{\text{start}})) \notin E_0$, and the restricted observer behavior $O(E_0)$ as the corresponding set of trajectories $(u, y, \hat{x})$.

We now call system (2) an *asymptotic observer* for system (1) if $\lim_{t\to\infty} d(x(t), \hat{x}(t)) = 0$ for all corresponding trajectories $(u, y, x) \in P$ and $(u, y, \hat{x}) \in O(E_0)$. Following Trumpf et al. (2014), we call the observer *nonintrusive* if for every plant trajectory $(u, y, x) \in P$ there exists at least one corresponding observer trajectory $(u, y, \hat{x}) \in O(E_0)$.

1.3 The internal model principle

We can now formulate the main result of this paper.

Theorem 1. Assume that system (1) is repeatable and that $\pi\colon L \longrightarrow M$ is a surjective fiber bundle with compact fibers. Let system (2) be a nonintrusive asymptotic observer for system (1) then $P \subset O$.

Intuitively, what this result says is that although we are only aiming to asymptotically observe the plant trajectories, i.e. approximately match them at large time, the observer nevertheless must be capable of producing the exact plant trajectories at all times. We will discuss an algebraic consequence for the right hand side of the observer further below but it should already be clear at this point that this result provides some form of lower limit on the required complexity of asymptotic observers. Our proof actually shows the slightly stronger result $P \subset O(E_0)$ but this is somewhat less elegant.

It can be shown that π being surjective is necessary in the sense that if π is a submersion that is not surjective then system (2) cannot be an asymptotic observer as long as P contains a trajectory that leaves the image of π. Compactness of the fibers is necessary as the following example shows.

Example 2. The system

$$\begin{aligned}\dot{z}_1 &= u - (z_1 - y) + \mathrm{e}^{-z_2},\\ \dot{z}_2 &= |z_2| + 1,\\ \hat{x} &= z_1\end{aligned}$$

on $L = \mathbb{R}^2$ is a nonintrusive asymptotic observer for the plant

$$\begin{aligned}\dot{x} &= u,\\ y &= x\end{aligned}$$

on $M = \mathbb{R}^1$ with π the projection on the first coordinate and $E_0 = \emptyset$. While O does not contain any of the trajectories in P in this example, it contains a copy of P that has been shifted by a fixed small (asymptotically zero) trajectory. Making this observation precise is the topic of ongoing research.

In order to re-formulate Theorem 1 in terms of conditions on the right hand side of equation (2), we recall that a family of vector fields $F\colon L \times U \longrightarrow TL$ on L is called a *lift* of the family $f\colon M \times U \longrightarrow TM$ on M via $\pi\colon L \longrightarrow M$ if $\mathrm{D}\pi(z)[F(z, u)] = f(\pi(z), u)$ for all $z \in L$ and $u \in U$. Given an Ehresmann connection H for π and its connection form $\omega\colon TL \longrightarrow TL$, the unique lift F of f defined by H fulfils $\omega \circ F = 0$, see del Hoyo (2016). It is easy to see that $\tilde{f} := (\mathrm{id}_{TL} - \omega) \circ \hat{f}$ fulfils $\tilde{f}(z, u, y) \in H(z)$ for all $z \in L$, $u \in U$ and $y \in Y$. Moreover, if H is complete and if we replace $\hat{f}$ in system (2) with $\tilde{f}$, this does not change the observer behavior O. This observation allows to prove the following algebraic version of Theorem 1.

Corollary 3. Assume that system (1) is repeatable and that $\pi\colon L \longrightarrow M$ is a surjective fibre bundle with compact fibers. Choose a complete Ehresmann connection H for π with connection form ω and let F be the lift of f defined by H. Let system (2) be a nonintrusive asymptotic observer for system (1) with observer behavior O. Then the modified observer

$$\begin{aligned}\dot{z} &= (\mathrm{id}_{TL} - \omega) \circ \hat{f}(z, u, y),\\ \hat{x} &= \pi(z),\end{aligned} \tag{3}$$

has the same observer behavior O, and there exists a family of vector fields $\Delta\colon L \times U \times Y \longrightarrow TL$ such that

$$(\mathrm{id}_{TL} - \omega) \circ \hat{f}(z, u, y) = F(z, u) + \Delta(z, u, y)$$

for all $z \in L$, $u \in U$ and $y \in Y$, and such that $\Delta \equiv 0$ along all trajectories (u, y, z) of system (3) with $(u, y, \pi(z)) \in P$.

In other words, the modified observer system (3) is of the form lift F of the plant plus correction term Δ that is zero along lifted plant trajectories, and retains the same observer behavior O as the original observer. This result shows that the generalized Luenberger construction for observers is indeed most general in this setting, in the sense that all possible asymptotically stable error behaviors can be achieved by such designs. An example of such a design is the construction of observers for kinematic systems with symmetry on the symmetry Lie group studied in Mahony et al. (2013) and Trumpf (2018).

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