Convergence and State Reconstruction of Time-varying Multi-agent Systems from Complete Observability Theory

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***Abstract*—We study continuous-time consensus dynamics for multi-agent systems with undirected switching interaction graphs. We establish a necessary and sufficient condition for exponential asymptotic consensus based on the classical theory of complete observability. The proof is remarkably simple compared to similar results in the literature and the conditions for consensus are mild. This observability-based method can also be applied to the case where negatively weighted edges are present. Additionally, as a by-product of the observability based arguments, we show that the nodes' initial value can be recovered from the signals on the edges up to a shift of the network average.**

I. Introduction

A. Motivation

Distributed consensus algorithms [1]–[4] of multi-agent systems have become the foundation of many distributed solutions to in-network control and estimation [5], [6], signal processing [7], and optimization [8], [9]. Convergence conditions of consensus dynamics often play a key role in the performance of such distributed designs, which are relatively well understood under discrete-time network dynamics when the underlying interaction graph is either time-invariant [2], [3], [14] or time-varying [11], [13].

Continuous-time consensus dynamics, however, is difficult to analyze with switching interaction graphs. For piece-wise constant graph signals with dwell time, we can treat the continuous-time flow by investigating the node state evolution along a selected sequence of time instants and then work on the discrete-time evolution at the sampled times [1], [4]. For graphs that vary in structure and edge weights in a general manner, the problem becomes much more challenging. Under such circumstances the maximum node state difference serves as one major tool for convergence analysis [12], [15], [20].

In this paper, we study the convergence of continuous-time consensus dynamics with undirected and switching interaction graphs from the classical theory on uniformly complete observability of linear time-varying systems [17], [19]. It turns out that the complexity of the convergence analysis can be drastically reduced in view of these classical results even under mild conditions. Interestingly enough, this observability-based method can also be used to analyze consensus dynamics over networks with both positively and negatively weighted edges [21], [22], where methods based on maximum node state difference function are no longer applicable.

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B. The Model

Consider a multi-agent network with N nodes indexed in the set $\mathbf{V} = \{1, \dots, N\}$. Each node i holds a state $x_i(t) \in \mathbb{R}$ for $t \in \mathbb{R}^{\geq 0}$. The dynamics of $x_i(t)$ are described by

$$\frac{d}{dt}x_i(t) = \sum_{j=1}^{N} a_{ij}(t)\big(x_j(t) - x_i(t)\big), \tag{1}$$

where the $a_{ij}(t) = a_{ji}(t)$ are, for the moment, assumed to be nonnegative numbers representing the weight of the edge between node i and j. For the weight functions $a_{ij}(t)$, we impose the following assumption as a standing assumption throughout our paper.

Weights Assumption. (i) Each $a_{ij}(t)$ is regulated in the sense that one-sided limits exist for all $t \geq 0$; (ii) The $a_{ij}(t)$ are almost everywhere bounded, i.e., there is a constant $A^* > 0$ such that $|a_{ij}(t)| \leq A^*$ for all $i, j \in \mathbf{V}$ and for almost all $t \geq 0$.

The underlying interaction graph of (1) at time t is defined as an undirected (weighted) graph $\mathbf{G}_t = (\mathbf{V}, \mathbf{E}_t)$, where $\{i, j\} \in \mathbf{E}_t$ if and only if $a_{ij}(t) > 0$. We introduce the following definition on the connectivity of the time-varying graph process $\{\mathbf{G}_t\}_{t\geq 0}$.

Definition 1: The time-varying graph $\{\mathbf{G}_t\}_{t\geq 0}$ is termed *jointly* (δ, T)*-connected* if there are two real numbers $\delta > 0$ and $T > 0$ such that the edges

$$\{i, j\} : \int_s^{s+T} a_{ij}(t)dt \geq \delta, \ i, j \in \mathbf{V} \tag{2}$$

form a connected graph over the node set $\mathbf{V}$ for all $s \geq 0$.

Along the system (1), it is obvious to see that the average of the node states

$$x_{\rm ave}(t) := \frac{\sum_{j=1}^{N} x_j(t)}{N}$$

is preserved over time. Therefore, if a consensus, i.e., all nodes asymptotically reaching a common state, is achieved, then all the node states will converge to $x_{\rm ave}(t_0)$ where $t_0 \geq 0$ is the system initial time. Denote by $\mathbf{1}_n$ the n-dimensional all one vector and $x(t) = (x_1(t) \cdots x_N(t))^{\mathrm{T}}$. We can therefore conveniently introduce the following definition.

Definition 2: Exponential asymptotic consensus is achieved if there are two constants $\alpha > 0$ and $\beta > 0$ such that

$$|x_i(t) - x_{\mathrm{ave}}(t_0)| \leq \beta \|x(t_0) - x_{\mathrm{ave}}(t_0)\mathbf{1}_N\| e^{-\alpha(t-t_0)}$$

for all $i \in \mathbf{V}$, all $x(t_0)$, and all $t \geq t_0$.

C. Main Results

First of all we present a necessary and sufficient condition for exponential asymptotic consensus of the system (1).

Theorem 1: Exponential asymptotic consensus is achieved for the system (1) if and only if there are $\delta > 0$ and $T > 0$ such that the graph $\{\mathbf{G}_t\}_{t\geq 0}$ is jointly (δ, T)-connected.

Theorem 1 is established in view of the classical complete observability theory for linear time-varying systems [18], [19]. A by-product of such observability-based analysis is a state reconstruction theorem, which indicates that node states can be recovered from the signals on the edges up to a shift of the network average. We recall the following definition.

Definition 3: ([17]) A linear time-varying system

$$\dot{x} = F(t)x \tag{3}$$
$$y = D^{\mathrm{T}}(t)x \tag{4}$$

is uniformly completely observable if there are some positive α_1, α_2 and δ such that

$$\alpha_1 I \leq \int_s^{s+\delta} \Phi^{\mathrm{T}}(t,s)D(t)D^{\mathrm{T}}(t)\Phi(t,s)dt \leq \alpha_2 I \tag{5}$$

for all $s \geq 0$. Here $\Phi(\cdot,\cdot)$ is the state transition matrix for system (3).

If the system (3) and (4) is uniformly completely observable, then

$$x(s) = \Big(\int_s^{s+\delta} \Phi^{\mathrm{T}}(t,s)D(t)D^{\mathrm{T}}(t)\Phi(t,s)dt\Big)^{-1} \int_s^{s+\delta} \Phi^{\mathrm{T}}(t,s)D(t)y(t)dt. \tag{6}$$

In this way, $x(s)$ is uniformly recovered from the signal $y(t) : t \in [s, s+\delta]$.

Theorem 2: Assume full knowledge of $\{\mathbf{G}_t\}_{t\geq 0}$ with the edge weights. The shifted node state $x_i(s) - x_{\mathrm{ave}}(0)$, $i \in \mathbf{V}$ can be uniformly recovered from the signals $x_j(t) - x_i(t)$, $\{i,j\} \in \mathbf{E}_t$ over the edges if there are two constants $\delta > 0$ and $T > 0$ such that the graph $\{\mathbf{G}_t\}_{t\geq 0}$ is jointly (δ, T)-connected.

D. Generalizations

We now present a few direct generalizations of the above established convergence results.

1) Robust Consensus: Further consider the system (1) subject to noises:

$$\frac{d}{dt}x_i(t) = \sum_{j=1}^N a_{ij}(t)\big(x_j(t) - x_i(t)\big) + w_i(t),\ i \in \mathbf{V}, \tag{7}$$

where $w_i(t)$ is a piecewise continuous function defined over $\mathbb{R}^{\geq 0}$ for each $i \in \mathbf{V}$. We introduce the following notion of robust consensus.

Definition 4: The system (7) achieves $*$-bounded robust consensus if for all $w(t) := (w_1(t) \dots w_N(t))^{\mathrm{T}}$ satisfying

$$\int_s^{s+\zeta} w^{\mathrm{T}}(t)w(t)dt \leq B_0 \tag{8}$$

for some $\zeta > 0$, some $B_0 \geq 0$, and all $s \geq 0$, there exists $\mathcal{C}(\zeta, B_0) > 0$ such that

$$\|x_i(t) - x_{\mathrm{ave}}(t)\| \leq \mathcal{C}(\zeta, B_0),\ \forall i \in \mathbf{V}, \forall t \geq t_0 \tag{9}$$

when $x_1(t_0) = x_2(t_0) = \cdots = x_N(t_0)$.

The following robust consensus result holds.

Theorem 3: The system (7) achieves $*$-bounded robust consensus if and only if there are $\delta > 0$ and $T > 0$ such that the graph $\{\mathbf{G}_t\}_{t\geq 0}$ is jointly (δ, T)-connected.

2) Signed Networks: Up to now we have assumed that all the $a_{ij}(t)$ are non-negative numbers. In fact, the complete observability analysis can be easily extended to the case when some of the $a_{ij}(t)$ are negative. The motivation[1] of studying the effect of possibly negative $a_{ij}(t)$ comes from the modelling of misbehaved links in engineering network systems, as well as opinion dynamics over social networks with mistrustful interactions represented by negative edges [22].

Although conventionally, the graph Laplacian is considered only for graphs with non-negative edges, for any $t \geq 0$, we continue to define the (weighted) Laplacian of the graph $\mathbf{G}_t$ as

$$L_{\mathbf{G}_t} = D_{\mathbf{G}_t} - A_{\mathbf{G}_t}$$

with $A(\mathbf{G}_t) = [a_{ij}(t)]$ and $D(\mathbf{G}_t) = \mathrm{diag}(\sum_{j=1}^N a_{1j}(t), \dots, \sum_{j=1}^N a_{Nj}(t))$. Note that although the $a_{ij}(t)$ can now be negative, $L_{\mathbf{G}_t}$ remains symmetric with at least one zero eigenvalue with $\mathbf{1}_N$ being a corresponding eigenvector. Introduce the following assumption.

Negative-Link Assumption. The Laplacian $L_{\mathbf{G}_t}$ is positive semi-definite for all $t \geq 0$.

This negative-link assumption requires that the influence of the negative links can be reasonably overcome by the positive links for any $t \geq 0$. The following result holds.

Theorem 4: Consider the system (1) with possibly negative $a_{ij}(t)$. Assume the Negative-Link Assumption. Then Theorems 1 and 3 continue to hold.

[1]Another definition for a negative link between i and j is to replace x_j by $-x_j$ in the system (1) [23]. The difference of the two definitions of negative links was discussed in [22] from a social network point of view.

E. Some Remarks

Theorems 1 and 3 are based on a direct application of the classical complete observability theory for linear time-varying systems [18], [19], after some useful properties of the Laplacian of the time-varying network have been established. The theorems recover the results in [1], [15], [20] for undirected graphs under weaker weight and noise assumptions. We would also like to emphasize that the applicability of the complete observability theory relies heavily on undirected node interactions, and hence we believe that it will be difficult to extend the analysis to the directed case.

Moreover, Theorem 4 illustrates that if the effect of the negative links is somewhat moderated by the positive links, i.e., as in the Negative-Link Assumption, then the convergence property is not affected by the negative links. This point, however, cannot be seen from the max state difference analysis used in prior work [15], [20] because obviously the max state difference function is no longer non-increasing in the presence of negative links, regardless of how small their weights are compared to the positive ones.

Theorem 2 discusses the possibility of reconstructing node initial states from signals over the edges. Naturally one may wonder whether we can recover the nodes' initial values by observing the state signals at one or more nodes. By duality this is equivalent to the controllability of the network by imposing control inputs over one or more anchor nodes. This problem, as shown in [24], [25], is challenging in general for a full theoretical characterization on general graphs, and in fact, finding the minimum number of nodes to control to ensure network controllability is a computationally intensive problem for general network dynamics [26].

II. Proofs of Statements

A. Proof of Theorem 1

Consider the following linear time-varying system

$$\dot{x} = -V(t)V^{\mathrm{T}}(t)x, \tag{10}$$

where $x \in \mathbb{R}^n$ and $V(\cdot)$ is a piecewise continuous and almost everywhere bounded matrix function mapping from $\mathbb{R}^{\geq 0}$ to $\mathbb{R}^{n\times r}$. Let $\Phi(\cdot,\cdot)$ be the state transition matrix of the System (10). Our proof relies on the following result which was proved using complete observability theory in [18].

Lemma 1: System (10) is exponentially asymptotically stable in the sense that there are $\gamma_1, \gamma_2 > 0$ such that $\|\Phi(t,s)\| \leq \gamma_1 e^{-\gamma_2(t-s)}$ for all $t \geq s \geq 0$, if and only if for some positive α_1, α_2 and δ, there holds

$$\alpha_1 I \leq \int_s^{s+\delta} V(t)V^{\mathrm{T}}(t)dt \leq \alpha_2 I \tag{11}$$

for all $s \in \mathbb{R}^{\geq 0}$.

Remark 1: To serve the technical purpose of the proof, we have made a slight variation in Lemma 1 compared to the original Theorem 1 in [18], where we have replaced the lower bound on exponentially asymptotic stability required in [18] with the almost everywhere boundedness of $V(\cdot)$ adopted in the current paper.

The system (1) can be written into the following compact form

$$\dot{x} = -L_{\mathbf{G}_t}x. \tag{12}$$

Since $L_{\mathbf{G}_t}$ is positive semi-definite for all $t \geq 0$ [16], we can easily construct $V(t)$ so that the system (12) can be re-written into the form of (10). Although any construction of $V(t)$ would do, we use the incidence matrix $H_{\mathbf{G}_t}$. We make the following definition.

Definition 5: The $N \times \frac{N(N-1)}{2}$ weighted incidence matrix of $\mathbf{G}_t$, denoted as $H_{\mathbf{G}_t}$, with rows indexed by the nodes $v \in \mathbf{V}$ and columns indexed by the edges $\{i,j\}$ (abbreviated as $\{ij\}$), $i<j \in \mathbf{V}$, is defined as $H_{\mathbf{G}_t} = [H_{v-\{ij\}}(t)]$ where[2]

$$H_{v-\{ij\}}(t) = \begin{cases} -\sqrt{a_{ij}(t)}, & \text{if } v=i; \\ \sqrt{a_{ij}(t)}, & \text{if } v=j; \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

We can now write (12) as

$$\dot{x} = -L_{\mathbf{G}_t}x = -H_{\mathbf{G}_t}H_{\mathbf{G}_t}^{\mathrm{T}}x. \tag{14}$$

Define $y(t) = (I - \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)x(t)$. Then we have

$$\begin{aligned}\dot{y} &= (I - \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)\dot{x}(t) \\ &= -(I - \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)L_{\mathbf{G}_t}x(t) \\ &\overset{a)}{=} -(L_{\mathbf{G}_t} + \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)(I - \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)x(t) \\ &= -(L_{\mathbf{G}_t} + \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)y(t) \\ &\overset{b)}{=} -\Big(H_{\mathbf{G}_t} + \frac{\mathbf{1}_N\mathbf{1}_r^{\mathrm{T}}}{\sqrt{rN}}\Big)\Big(H_{\mathbf{G}_t} + \frac{\mathbf{1}_N\mathbf{1}_r^{\mathrm{T}}}{\sqrt{rN}}\Big)^{\mathrm{T}} y(t),\end{aligned} \tag{15}$$

where $r = N(N-1)/2$, $a)$ follows from the equations that $(I - \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)L_{\mathbf{G}_t} = L_{\mathbf{G}_t}(I - \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)$ and $(\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)^2 = \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N$, and $b)$ follows from direct computation in view of $L_{\mathbf{G}_t} = H_{\mathbf{G}_t}H_{\mathbf{G}_t}^{\mathrm{T}}$.

We can now conclude that the following statements are equivalent.

(i) System (12) achieves exponential consensus;
(ii) System (15) is exponentially asymptotically stable;
(iii) There are some positive α_1, α_2 and T, such that

$$\begin{aligned}\alpha_1 I &\leq \int_s^{s+T} \Big(H_{\mathbf{G}_t} + \frac{\mathbf{1}_N\mathbf{1}_r^{\mathrm{T}}}{\sqrt{rN}}\Big)\Big(H_{\mathbf{G}_t} + \frac{\mathbf{1}_N\mathbf{1}_r^{\mathrm{T}}}{\sqrt{rN}}\Big)^{\mathrm{T}} dt \\ &= \int_s^{s+T} (L_{\mathbf{G}_t} + \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)dt \leq \alpha_2 I;\end{aligned} \tag{16}$$

for all $s \in \mathbb{R}^{\geq 0}$.
(iv) there are $\delta, T > 0$ such that the graph $\{\mathbf{G}_t\}_{t\geq 0}$ is (δ,T)-connected.

The equivalence of (i) and (ii) is straightforward. Lemma 1 gives the equivalence between (ii) and (iii). We only need to show the equivalence between (iii) and (iv).

(iii) $\Rightarrow$ (iv): If (16) holds, then $\int_s^{s+T}(L_{\mathbf{G}_t} + \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)dt$ is positive definite. Note that $\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N$ has a unique positive

[2] In this definition we have used an orientation for the graph $\mathbf{G}_t$ in which i is the tail of an edge $\{i,j\}$ if $i<j$. There always holds $L_{\mathbf{G}_t} = H_{\mathbf{G}_t}H_{\mathbf{G}_t}^{\mathrm{T}}$ independent of the choice of orientation in the definition of the incidence matrix $H_{\mathbf{G}_t}$ [16].

eigenvalue equal to 1 and another eigenvalue equal to zero with multiplicity $N-1$. Moreover, for any fixed $t \geq 0$, $L_{\mathbf{G}_t}\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N = \big(\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N\big)L_{\mathbf{G}_t} = 0$, and the zero eigenvalue of $L_{\mathbf{G}_t}$ and the eigenvalue one of $\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N$ share a common eigenvector $\mathbf{1}_N$. This implies that the second smallest eigenvalue of $\int_s^{s+T} L_{\mathbf{G}_t}dt$, $\lambda_2(\int_s^{s+T} L_{\mathbf{G}_t}dt)$, must satisfy $\lambda_2(\int_s^{s+T} L_{\mathbf{G}_t}dt) \geq \alpha_1$.

Consider $\lambda_2(L)$, the second smallest eigenvalue of L, where L takes value from

$$\mathscr{L} := \big\{L \in \mathbb{R}^{N\times N} : L \text{ is symmetric}; [L]_{ij} \leq 0, i \neq j; \\ L\mathbf{1}_n = 0\big\}.$$

Certainly $\lambda_2(\cdot)$ is a continuous function over the set $\mathscr{L}$. In turn, there must exist $\delta > 0$ (which depends on α_1) such that the edges

$$\{i,j\} : \int_s^{s+T} a_{ij}(t)dt \geq \delta, \ i,j \in \mathbf{V} \tag{17}$$

form a connected graph over the node set $\mathbf{V}$. In other words, the graph $\{\mathbf{G}_t\}_{t\geq 0}$ is (δ, T)-connected.

(iv) $\Rightarrow$ (iii): The matrices $\int_s^{s+T} L_{\mathbf{G}_t}dt$, for all $s \geq 0$ and all (δ,T)-connected graphs $L_{\mathbf{G}_t}$, form a compact set in $\mathscr{L}$. Therefore, again noticing that $\lambda_2(\cdot)$ is a continuous function over the set $\mathscr{L}$, we can find a constant $\alpha_1 > 0$ (which depends on δ) such that $\lambda_2(\int_s^{s+T} L_{\mathbf{G}_t}dt) \geq \alpha_1$ for all $s \geq 0$ and for all (δ,T)-connected graph $\{\mathbf{G}_t\}_{t\geq 0}$. Re-using the above properties of the two matrices $L_{\mathbf{G}_t}$ and $\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N$ we immediately get the lower bound part of (16). The existence of α_2 in (16) is always guaranteed by the Weights Assumption on the almost everywhere boundedness of the $a_{ij}(t)$.

We have now proved Theorem 1 by the equivalence of (i) and (iv).

B. Proof of Theorem 2

From the proof of Theorem 1, if there are $\delta, T > 0$ such that the graph $\mathbf{G}(t)$ is (δ,T)-connected, then there are some positive α_1, α_2 and T, there holds

$$\alpha_1 I \leq \int_s^{s+T} \Big(H_{\mathbf{G}_t} + \frac{\mathbf{1}_N\mathbf{1}_r^{\mathrm{T}}}{\sqrt{rN}}\Big)\Big(H_{\mathbf{G}_t} + \frac{\mathbf{1}_N\mathbf{1}_r^{\mathrm{T}}}{\sqrt{rN}}\Big)^{\mathrm{T}} dt \leq \alpha_2 I; \tag{18}$$

for all $s \geq 0$. This in turn implies that the following system (see [18])

$$\begin{cases} \dot{y} &= -(L_{\mathbf{G}_t} + \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)y; \\ z &= \Big(H_{\mathbf{G}_t} + \frac{\mathbf{1}_N\mathbf{1}_r^{\mathrm{T}}}{\sqrt{rN}}\Big)^{\mathrm{T}} y. \end{cases} \tag{19}$$

is uniformly completely observable.

Note that

$$\begin{aligned} z(t) &= \Big(H_{\mathbf{G}_t} + \frac{\mathbf{1}_N\mathbf{1}_r^{\mathrm{T}}}{\sqrt{rN}}\Big)^{\mathrm{T}} y(t) \\ &= \Big(H_{\mathbf{G}_t}^{\mathrm{T}} + \frac{\mathbf{1}_r\mathbf{1}_N^{\mathrm{T}}}{\sqrt{rN}}\Big)(I - \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)x(t) \\ &= H_{\mathbf{G}_t}^{\mathrm{T}} x(t). \end{aligned} \tag{20}$$

This immediately translates to the desired theorem.

C. Proof of Theorem 3

Again we consider $y(t) = (I - \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)x(t)$. Along the system (7) the evolution of $y(t)$ obeys

$$\frac{d}{dt}y(t) = -(L_{\mathbf{G}_t} + \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)y(t) + (I - \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)w(t). \tag{21}$$

It is then obvious that system (7) achieves $*$-bounded robust consensus under Definition 4 if system (21) is bounded*-input, bounded state stable as defined in [17].

If the graph $\{\mathbf{G}_t\}_{t\geq 0}$ is (δ,T)-connected, then (15) is exponentially asymptotically stable. This in turn leads to the conclusion that the system (21) is bounded*-input, bounded state stable based on the sufficiency part of Theorem 1 (p. 404) in [17]. Therefore, system (7) achieves $*$-bounded robust consensus.

On the other hand, note that there holds

$$(L_{\mathbf{G}_t} + \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N)(I - \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N) = (L_{\mathbf{G}_t} + \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N) \tag{22}$$

for all t. Consequently, we have

$$\Phi(t,s)(I - \mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}/N) = \Phi(t,s) \tag{23}$$

for all t and s, where $\Phi(t,s)$ is the state-transition matrix of the system (21) with $w(t) \equiv 0$. Therefore, applying the necessity proof of Theorem 1 in [17] we can directly conclude that if system (21) is bounded*-input, bounded state stable, then (15) is exponentially asymptotically stable. This in turn leads to the graph $\{\mathbf{G}_t\}_{t\geq 0}$ being (δ,T)-connected.

We have now completed the proof.

D. Proof of Theorem 4

When the Negative-Link Assumption holds, the construction of (14) can be replaced by

$$\dot{x} = -L_{\mathbf{G}_t}x = -\sqrt{L_{\mathbf{G}_t}}\sqrt{L_{\mathbf{G}_t}}x. \tag{24}$$

It is then straightforward to see that the remaining arguments in the proofs of Theorems 1 and 3 will not be affected, leading to the desired result.

III. Conclusions

We have studied continuous-time consensus dynamics with undirected switching interaction graphs in view of the classical theory of uniform complete observability for linear time-varying systems. A necessary and sufficient condition for exponential asymptotic consensus was established by much simplified analysis compared to related results in the literature. Interestingly and importantly, this observability-based method can also be applied to signed networks with both positive and negative edges. We also established a robust consensus result as well as a state reconstruction result indicating that the initial values can be recovered from signals on the edges up to a shift of the network average.

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