Recursive Attitude Estimation in the Presence of Multi-rate and Multi-delay Vector Measurements

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Abstract— This paper proposes an attitude estimation methodology for the case where attitude sensors provide discrete-time samples of vector measurements at different sample rates and with time delays. The proposed methodology is based on a cascade combination of an output predictor and an attitude observer or filter. The predictor compensates for the effect of sampling and delays in vector measurements and provides continuous-time predictions of outputs. These predictions are then used in an observer or filter to estimate the current attitude. The primary contribution of the paper is to exploit the underlying symmetry of the attitude kinematics to design a recursive predictor that is computationally simple and generic, in the sense that it can be combined with any asymptotically stable observer or filter. We prove that the predictor is able to reproduce the continuous time delay-free vector measurements. In a simulation example, we demonstrate good performance of the combined predictor-observer even in presence of measurement noise and delay uncertainties.

I. INTRODUCTION

Attitude sensors mounted on a vehicle measure partial information about its attitude in the form of vector direction measurements. The goal of an attitude estimator is to compute the orientation of the vehicle by processing those vector measurements. There is a large body of research on both stochastic attitude estimation methods (such as extended Kalman filters [1], [2], unscented filters [3], etc.) as well as deterministic attitude observers [4]–[16]. In satellite attitude estimation applications, high accuracy sensors such as star trackers or earth sensors provide measurements at low sampling rates (0.5 to 10 Hz) [17]. In contrast, the onboard gyroscope can easily provide high bandwidth measurements at kHz rates, potentially two orders of magnitude faster than the direction information is obtained. The image processing inside a star-tracker sensor can cause significant delays in the order of tens of milliseconds, leading to the star-tracker measurement being delayed with respect to the gyroscope measurements. Similar sampling and delay problems also occur in attitude estimation for aerial robots when vision based sensors such as cameras and landmarks are employed. Also, in indoor flight environments, the attitude data from devices such as VICON or OptiTrack are delayed by the communication channel from these sensors to the onboard attitude estimation system of the vehicle.

Sampling and delays can negatively affect the stability and robustness of any observer or filter and degrade their performance if they are not compensated for properly [18]–[24]. Typical estimator design methodologies to tackle the measurement sampling and delay problem are; estimator design with Lyapunov-Krasovskii modification, stochastic filtering with Out-Of-Sequence Measurements (OOSM), and compound observer-predictor design. The classical approach to tackle the sensor delay is to take an estimator that has the desired performance for delay free measurements, and modify its innovation term such that it compares each delayed measurement with its corresponding backward time-shifted estimate. If the delay-free estimator has a Lyapunov stability proof, the stability analysis for the modified estimator can be undertaken using Lyapunov-Krasovskii functions [25], [26]. Although these modified estimators are commonly used in practice (see e.g. [27], [28]), they require complicated stability analyses and careful and conservative gain tuning, leading to poor transient responses of the resulting estimators. Stochastic filtering with OOSM has been extensively studied [29]–[33], albeit most of this literature focuses on target tracking applications. Although OOSM filtering approaches are flexible, easily dealing with sampled and delayed data as well as out-of-sequence measurements, they usually have significant memory and processing requirements that are unrealistic for most embedded observer design applications, except for linear system models where simpler OOSM filters are available [20], [29]. For the specific problem of attitude estimation with sampled and delayed measurements, a modified extended Kalman filter with a novel real time implementation architecture is proposed in [34]. Despite its good performance in practice, this algorithm suffers from major drawbacks such as unclear convergence properties and high computational load due to the required propagation stages associated with sensor delay compensation. Combined observer-predictor design methods for nonlinear systems on $\mathbb{R}^n$ have been developed in [19], [24], [35]. These methods take observers that have the desired stability properties for continuous delay-free measurements and combine them with appropriate predictors that compensate for the effects of sensor sampling and delays, such that the combined observer-predictor maintains the stability properties of the observer. The authors of this paper have recently proposed a cascade observer-predictor combination to handle sensor delay in the attitude estimation problem [36]. Although the resulting observer-predictor combination is stable, this method

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requires continuous availability of sensor outputs and is not applicable to the sampled measurement case. To the authors' knowledge, there is no attitude estimation methodology with stability proof available that considers sampled and delay measurements.

In this paper, we consider the attitude estimation problem when sampled and delayed vector measurements are available. We propose a cascade combination of a predictor with an attitude observer or filter in which the predictor compensates for the effect of sampling as well as delays in vector measurements and the filter or observer processes the predicted outputs and estimates the attitude. Our design is based on the exact continuous time nonlinear attitude kinematics on the Lie group SO(3) without resorting to parameterization, linearization, or discretization. The main contribution of the paper is to effectively employ the symmetries of the attitude kinematics and vector measurement models to design a simple generic predictor that is independent of the choice of observer or filter. That is, our proposed predictor can be combined with any observer or filter that has asymptotically stable estimation error in ideal conditions (i.e. when it is fed with continuous time delay-free vector measurements) and the predictor-observer combination maintains those stability properties in the non-ideal conditions (i.e. sampled and delayed measurements). The proposed predictor is recursive and requires only very small computational power, making it ideal for embedded implementation in real-world applications. We assume that the delay in each sensor measurement is known, that is we require accurate time-stamping of data, however, this is the only condition on the data. Given this assumption, the gain tuning process and the stability of the observer is independent of the size of the delay and valid even for time varying delays or OOSM measurements although we do not explicitly consider the latter in this paper. The proposed approach directly extends to the multi-rate measurement case without further modification. Via a simulation example, we show that our predictor-observer method performs significantly better than Lyapunov-Krasowskii approach.

The structure of the paper is as follows. Background and problem formulation is given in section II. The proposed predictor-observer approach is described in section III where the main result of the paper is given by Theorem 1. The performance of our method is demonstrated via simulations in section IV.

II. PROBLEM FORMULATION

Attitude determination sensors aim to measure physical quantities that are usually continuous time objects by their nature. Examples of these physical quantities are the light intensity of stars or the Sun, respectively, sensed by star sensors or sun sensors, the Infra-Red reflection of the Earth surface sensed by Earth sensors, or the magnetic field of the Earth sensed by magnetometers, all of which are continuous time objects. In practice, however, attitude sensors are only capable of providing samples of those physical quantities at specific sampling rates. Moreover, these samples are usually delayed with respect to the measured physical quantities due to various reasons such as slow response rates of the physical parts of the sensors, internal processing time of sensors, and communication delays. In the following two subsections, we present a general discussion about modeling sampling and delays in sensors. This discussion will then be applied to the specific case of attitude sensors and vector measurements in Section II-C.

A. Physically inspired modeling of sampling and delays

We propose the model illustrated in Fig. 1 to include the effect of sampling and delays on the output of sensors. This model is inspired by the physical process that takes place in sensors during measuring a physical quantity. This model consists of a zero-order-hold (ZOH) block that models the effect of sampling and two delay blocks before and after the ZOH that, respectively, model the pre-sampling and post-sampling delays. The pre-sampling delay on the left side of Fig. 1 models ρ_i seconds of delay from when the physical quantity $y_i(t)$ occurs to when it is observed by the i-th sensor. We have $y_i^\rho(t) = y_i(t - \rho_i)$ for all t. In practice, this delay is usually due to the physical properties of the environment or the sensors. For instance, a star tracker requires that its imaging sensor is exposed to light from stars for a specific amount of time so that it can produce an image of the stars. This is known as exposure time and can be as large as hundreds of milliseconds [37]. The ZOH block in Fig. 1 takes the delayed signal and produces a sample at time t_{k_i}. This sample is latched at the output of ZOH until the next sample is taken at time t_{k_i+1}. Hence we have $z_i^\rho(t) = y_i^\rho(t_{k_i}) = y_i(t_{k_i} - \rho_i)$ for $t \in [t_{k_i}, t_{k_i+1})$. For clarity in presentation, we assume that the sequence $(t_{k_i})_{k_i=1}^\infty$ is an ordered monotonically increasing sequence, i.e. $t_{k_i-1} \leq t_{k_i} \leq t_{k_i+1}$. However, this assumption is not necessary for our proposed method and our method is also applicable to the case where the measurements are out-of-sequence, although the necessary modifications to the notation are rather cumbersome. For a star tracker, the sequence $(t_{k_i})_{k_i=1}^\infty$ corresponds to the specific times when the star tracker obtains an image of stars. This sampling frequency can be as low as only 0.5 Hz up to 10 Hz for practical star trackers. The post-sampling delay on the right side of Fig. 1 models σ_i seconds of delay from when a sample of the physical variable becomes available to the sensor to the time when the new output $z_i(t)$ becomes available to the user. Hence we have

$$z_i(t) = z_i^\rho(t - \sigma_i) = y_i(t_{k_i} - \rho_i), \ t \in [t_{k_i} + \sigma_i, t_{k_i+1} + \sigma_i) \tag{1}$$

In practice, the post-sampling delay models the delay due to the internal signal processing in the sensor or due to the communication delay for transmitting information from the sensor to the user. For a star tracker, the post-sampling delay is mainly due to the processing time associated with image processing algorithms that analyze the images taken by the star tracker to recognize stars in the image and associate each recognized star with its corresponding star in the on-board
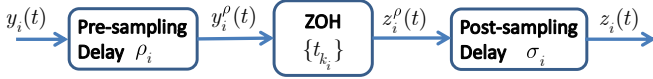
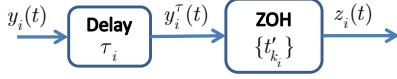Fig. 1. Modelling the effect of sampling and delays in attitude sensors

Fig. 2. Simple model that is input-to-output equivalent to Fig. 1

star catalog. The post-sampling delay can also model the lag due to the communication delay from VICON or OptiTrack systems to the onboard attitude estimation system of flying vehicles in indoor flight environments. It also can model the measurement delay associated with internal processing in GPS modules.

B. An input-to-output equivalent model

Although the model illustrated in Fig. 1 is suitable at the modeling stage to carefully describe the effect of sampling and various delays, it might not be convenient to be employed for compensation of delays and sampling effects at the design stage. The main disadvantage of this model is that even if the user knows the value of σ_i, the value of t_{k_i} is only available from the time $t_{k_i} + \sigma_i$ onwards. That is, the sensors usually do not inform the user when exactly they obtain samples of measurements. Instead, they inform the user when they finish processing the samples and the processing result (i.e. the sensor output) is ready to be collected by the user. This is why we discuss a simpler model here which is input-to-output equivalent to Fig. 1, but is more convenient to use for compensation of the effect of sampling and delays. Later on in this section, we discuss the condition under which these two models are equivalent.

Assume that at time t'_{k_i}, $k_i = 1, 2, \ldots$, we receive the most recent output of the i-th sensor denoted by $z_i(t'_{k_i})$. We assume that this output is delayed τ_i seconds with respect to the measured physical quantity $y_i(t'_{k_i})$. This output is latched until the next output arrives at t'_{k_i+1}. This procedure is equivalent to a cascade combination of a delay operator and a ZOH, as sketched in Fig. 2. We have

$$z_i(t) = z_i(t'_{k_i}) = y_i(t'_{k_i} - \tau_i), \ t \in [t'_{k_i}, t'_{k_i+1}). \quad (2)$$

Each sequence $(t'_{k_i})_{k_i=1}^{\infty}$ is ordered monotonically increasing (again, this assumption is not necessary for our method but it is imposed for clarity in presentation). The main difference compared to the model discussed in Section II-A is that here the sequence (t'_{k_i}) is known to the user and can be used for compensating the effect of sampling and delays. It is obvious that the outputs of Fig. 1 and Fig. 2 can be different in general. However, simple calculations provided in Appendix A show that both models are input-to-output equivalent (i.e. the output $z_i(t)$ of both models are equal for all t when both models measure the same physical quantity $y_i(t)$) if and only if $\tau_i = \rho_i + \sigma_i$ and $(t'_{k_i})_{k_i=1}^{\infty} = (t_{k_i} + \sigma_i)_{k_i=1}^{\infty}$. That is, when both models are equivalent, the delay τ_i in Fig. 2 represents

the combination of the delays ρ_i and σ_i of Fig. 1 and the sampling sequence $(t'_{k_i})_{k_i=1}^{\infty}$ in Fig. 1 is equivalent to the sequence of times by which the user receives the outputs $z_i(t)$ of Fig. 2. Given a model in the form of Fig. 1, we are always able to simplify that model to the form of Fig. 2 by proper choice of the delay and sampling sequence in Fig. 2. This in particular means that, as far as the input-to-output characteristics of the sensors are concerned, there is no need to separately know the value of the pre-sampling and post-sampling delays. In fact, as we show in Section III, only the knowledge of the total delay τ_i between the physical value $y_i(t)$ and the sensor output suffice to reproduce the physical quantity $y_i(t)$ from the sampled and delayed sensor output.

C. Attitude kinematics and sensor models for vector measurements

Consider a rigid body with a body-fixed reference frame $\{B\}$ and an inertial reference frame $\{A\}$. Denote the attitude matrix of the rigid body by $R \in \mathrm{SO}(3)$ that corresponds to the rotation from $\{B\}$ to $\{A\}$. The rigid body attitude kinematics is given by

$$\dot{R}(t) = R(t)\Omega(t)_\times, \ R(0) = R_0 \quad (3)$$

where Ω is the angular velocity vector of $\{B\}$ with respect to $\{A\}$ expressed in $\{B\}$. The linear operator $(.)_\times$ maps any vector in $\mathbb{R}^3$ to its corresponding skew-symmetric matrix in $\mathfrak{so}(3)$ such that $(a)_\times b$ is equal to the cross product $a \times b$ for all $a, b \in \mathbb{R}^3$. We assume that delay-free measurements of $\Omega(t)$ are available in continuous time. This is a reasonable assumption since in practice $\Omega(t)$ is measured at a high sampling rate using 3-axis gyros.

Ideal attitude sensors attached to the rigid body provide partial measurements of attitude in the form of vector measurements given by

$$y_i(t) = R(t)^\top \mathring{y}_i, \ i = 1, 2, \ldots, n \quad (4)$$

where $y_i \in \mathrm{S}^2$ denotes the measured vector in $\{B\}$ and $\mathring{y}_i \in \mathrm{S}^2$ denotes the corresponding reference vector of y_i in $\{A\}$. One can replace $y_i(t)$ from (4) into (1) and (2) to obtain attitude sensor models with sampling and delays corresponding to Fig. 1 and Fig. 2, respectively.

The problem at hand is to design an estimation methodology that uses the continuous measurement of $\Omega(t)$ together with the sampled and delayed vector measurements $z_i(t)$ to provide continuous estimates of the attitude matrix $R(t)$.

III. PREDICTOR-OBSERVER APPROACH

Due to the reasons discussed in Section II-B, we opt to work with the simplified sensor model (2) to design an algorithm that compensates for the effects of sampling and both pre and post-sampling delays and estimates the attitude. The approach that we propose here to tackle the problem formulated in Section II is illustrated in Fig. 3. We first propose a predictor that takes the sampled and delayed measurements $z_i(t)$ and provides continuous time predictions of $y_i(t)$ denoted by $y_i^p(t)$. The predictor relies on the knowledge of $\Omega(t)$ in continuous time (or practically at high frequency)

$$\Omega(t),\ z_i(t_{k_i}),\ \tau_i,\ i=1,...,n$$

Predictor

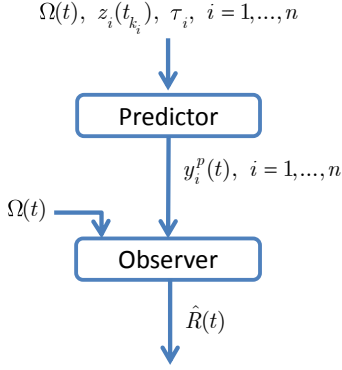$$y_i^p(t),\ i=1,...,n$$

$$\Omega(t)$$

Observer

$$\hat{R}(t)$$

Fig. 3. Illustration of the proposed predictor-observer approach (5)-(7)

and the total delay $\tau_i = \rho_i + \sigma_i$ to compensate for the effect of sampling and delay in the outputs and to predict the outputs such that $y_i^p(t) = y_i(t)$ for all $t \geq t'_{1_i}$ in noise-free conditions (i.e. when there is no measurement noise in $z_i(t'_{k_i})$ or $\Omega(t)$ and the integration procedure within the predictor is also exact). The predicted outputs $y_i^p(t),\ i=1,\ldots,n$ together with the angular velocity measurement are then fed into an observer to compute an estimate of attitude denoted by $\hat{R}(t)$. Our proposed predictor is generic in the sense that it is independent of the employed observer algorithm, i.e., the predictor can be coupled with any asymptotically stable attitude observer or filter to estimate the attitude.

Our proposed predictor takes the form

$$\dot{\Delta}(t) = \Delta(t)\Omega(t)_\times,\ \Delta(0) = \Delta_0, \tag{5}$$
$$y_i^p(t) = \Delta(t)^\top \Delta(t'_{k_i} - \tau_i)z_i(t),\quad t \in [t'_{k_i}, t'_{k_i+1}), \tag{6}$$

where $\Delta \in SO(3)$ is the internal state of the predictor and $\Delta_0 \in SO(3)$ is an arbitrary initial condition. The trajectory $\Delta(t)$ of the predictor dynamics (5) needs to be stored in a buffer for the previous $t'_{k_i+1} - t'_{k_i} + \tau_i$ seconds in order to compute the prediction $y_i^p(t)$ (6) at each time.

The following theorem summarizes the properties of the proposed predictor.

Theorem 1: Consider the predictor (5)-(6) for the attitude dynamics (3) and the sensor measurements (2) with (4). The predicted output $y_i^p(t)$ is equal to the ideal vector measurement $y_i(t)$ for all $t > t_{1_i}$ and all choices of $\Delta_0 \in SO(3)$. □

Proof of Theorem 1 is given in Appendix B.

Even though the proposed predictor is independent of the choice of observer, for the sake of concreteness, here we couple the predictor with the following geometric attitude observer [10].

$$\dot{\hat{R}}(t) = \hat{R}(t)\Big(\Omega(t) - P(t)\sum_{i=1}^{n}\big(L_i(\hat{y}_i(t) - y_i^p(t))\big)_\times \mathring{y}_i(t)\Big)_\times, \tag{7}$$

with $\hat{R}(0) = \hat{R}_0 \in SO(3)$ and $t \geq \max\limits_{i=1,\ldots,n} t_{1_i}$, where $\hat{R}(t)$ is the estimate of $R(t)$, $\hat{y}_i(t) := \hat{R}(t)^\top \mathring{y}_i$, and $P(t)$ and $L_i,\ i=1,\ldots,n$ are positive definite gain matrices. Stability

of the pure observer (7) is shown in [38] for general positive definite gain matrices $P(t)$ and $L_i,\ i=1,\ldots,n$. These gains are chosen to obtain the desired observer performance. For instance, $P(t)$ can be recursively updated using Riccati differential equations as in extended Kalman filters [1], [2], or it can be obtained using modified Riccati differential equations as in geometric approximate minimum-energy filters [10]. Choosing constant positive definite gain matrices simplifies (7) to the well-known geometric attitude observer proposed in [4], [7], [11], [39]. In any case, assuming that the observer (7) fed with the ideal vector measurements $y_i(t)$ rather than the predicted outputs $y_i^p(t)$ (i.e. replacing $y_i^p(t)$ with $y_i(t)$ in (7)) yields stable estimation error dynamics, then Theorem 1 implies that the combined predictor-observer (5),(6), and (7) retains those stability properties of the observer for all choices of $\Delta_0 \in SO(3)$. In particular, if the constant gain observer of [4] is employed as (7), then the estimated attitude $\hat{R}(t)$ converges almost globally asymptotically and locally exponentially to the true attitude $R(t)$ for all choices of $\Delta_0 \in SO(3)$.

Note that in order to implement the proposed predictor-observer methodology, it is only required to implement one copy of the predictor dynamics (5) and one copy of the observer dynamics (7) even though we have several vector measurements $z_i(t),\ i=1,\ldots,n$ with possibly different delays τ_i and possibly different sampling sequences $(t'_{k_i})_{k_i=1}^\infty$. Only a fixed duration buffer for the predictor state $\Delta(t)$ is needed.

Remark 1: Our proposed method is also applicable to the case where the delay τ_i is time-varying and out-of-sequence measurements do potentially occur. In this case, we should replace the notation τ_i with τ_{k_i} (forming the sequence $(\tau_{k_i})_{k_i=1}^\infty$) and each measurement delay τ_{k_i} should be known to the user at time t'_{k_i}. □

Remark 2: Although the predictor-observer idea presented in this paper focuses on the attitude kinematic system on $SO(3)$, this idea can be generalized to kinematic systems on general Lie groups. For the very special case where the underlying Lie group is $\mathbb{R}^n$, the kinematic system is simply the linear integrator $\dot{x}(t) = u(t)$ where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^n$ is the input. The output is given by $y(t) = Cx(t) \in \mathbb{R}^m$ where $C \in \mathbb{R}^{m \times n}$, and the sensors provide the delayed measurement $z(t) = y(t - \tau)$. It is easy to adapt the predictor proposed in this paper and obtain the following simple predictor

$$\dot{\delta}(t) = u(t), \tag{8}$$
$$y^p(t) = C(\delta(t) - \delta(t - \tau)) + z(t). \tag{9}$$

In this case, the predictor (8)-(9) corresponds to the well-known Smith predictor [40] originally designed for output feedback control of linear systems with delayed measurements. Note, however, that in the context of observers this predictor does not seem to suffer from the Smith predictor's well documented stability issues in the presence of delay uncertainty (see section IV). □

IV. SIMULATION RESULTS

In this section, we provide a set of simulations to illustrate the performance of our proposed predictor-observer methodology. To generate the trajectory of $R(t)$, we implement (3) with $\Omega(t) = [0;0;8]$ (deg/s) and the initial attitude R_0 corresponding to the initial roll 14 (deg), pitch 0 (deg), and yaw 0 (deg). We suppose that the attitude sensors provide the vector measurements corresponding to the reference directions $\mathring{y}_1 = [1\ 0\ 0]^\top$ and $\mathring{y}_1 = [0\ 1\ 0]^\top$. Although in practice the number of vector measurements can be high and their directions are not necessarily pairwise perpendicular (e.g. for star trackers), here we consider only two vector measurements with perpendicular directions to avoid unnecessary discussions on gain tuning and focus only on the sampling and delay effects. To model $z_1(t)$ and $z_2(t)$, the ideal vector measurements $y_1(t)$ and $y_2(t)$ are obtained by (4) and then fed to the block diagram of Fig. 1 with pre and post-sampling delays of $\rho_1 = \rho_2 = 0.1$ (s) and $\sigma_1 = \sigma_2 = 0.3$ (s), respectively, yielding a total delay of $\tau_1 = \tau_2 = 0.4$ (s), and a sampling rate of 5 (Hz). Zero mean Gaussian noises with a standard deviation of 0.01 are added to each axis of the vector measurements $z_1(t)$ and $z_2(t)$ which approximately add perturbations with the standard deviation of 1 (deg) to the directions of $z_1(t)$ and $z_2(t)$. The angular velocity $\Omega(t)$ is sampled at 100 (Hz) and perturbed by an additive noise of 0.05 (deg/s) in each axis.

For the simulation, we combine the predictor (5)-(6) with the geometric observer of [4]. This observer corresponds to choosing scalar constant observer gains in (7) yielding

$$\dot{\hat{R}}(t) = \hat{R}(t)\big(\Omega(t) + l_1 y_1^p(t) \times \hat{y}_1(t) + l_2 y_2^p(t) \times \hat{y}_2(t)\big)_\times \tag{10}$$

with $\hat{y}_i(t) := \hat{R}(t)^\top \mathring{y}_i(t)$ and $l_i > 0$, $i = 1, 2$. We compare the performance of this combined predictor-observer with an ad-hoc adaptation of the constant gain observer of [4] to the case of sampled and delayed vector measurements. The dynamics of the ad-hoc observer is given by $\dot{\hat{R}}_{ad}(t) = \hat{R}_{ad}(t)\big(\Omega(t) + \alpha(t)\big)_\times$ where $\hat{R}_{ad}(t)$ is the estimate of $R(t)$ and $\alpha(t)$ is the innovation term. When the attitude sensor provides the measured sample $z_1(t'_{k_1})$ at time $t = t'_{k_1}$, the innovation term of the ad-hoc observer is inspired by the constant gain observer as $\alpha(t'_{k_1}) = \bar{l}_1 z_1(t'_{k_1}) \times \hat{R}_{ad}(t'_{k_1} - \tau_1)^\top \mathring{y}_1$ with $\bar{l}_1 > 0$. This innovation term compares the newly received measurement $z_1(t'_{k_1})$ with its estimate $\hat{R}_{ad}(t'_{k_1} - \tau_1)^\top \mathring{y}_1$ in which the effect of the measurement delay τ_1 is considered[1]. Similarly, at time $t = t'_{k_2}$ when the measurement $z_2(t'_{k_2})$ is delivered by attitude sensors, the innovation term is $\alpha(t'_{k_2}) = \bar{l}_2 z_2(t'_{k_2}) \times \hat{R}_{ad}(t'_{k_2} - \tau_2)^\top \mathring{y}_2$ with $\bar{l}_2 > 0$. If t'_{k_2} happens to be equal to t'_{k_1} for some pair (k_1, k_2), then the innovation term is simply the sum $\bar{l}_1 z_1(t'_{k_1}) \times \hat{R}_{ad}(t'_{k_1} - \tau_1)^\top \mathring{y}_1 + \bar{l}_2 z_2(t'_{k_2}) \times \hat{R}_{ad}(t'_{k_2} - \tau_2)^\top \mathring{y}_2$. For the times where no sample of any vector measurement is available (i.e. for all $t \notin (t_{k_1})_{k_1=1}^\infty \cup (t_{k_2})_{k_2=1}^\infty$), the

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[1]Due to the consideration of the effect of delay in the innovation term, it can be thought of as a Lyapunov-Krasovskii term [25], [26], [35].
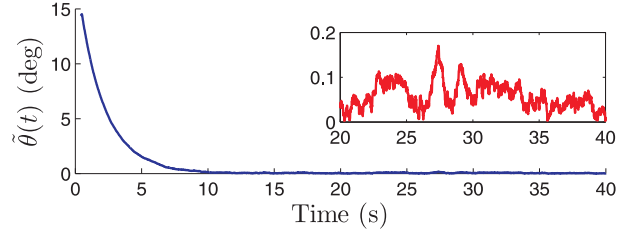

Fig. 4. Attitude estimation error of combined predictor-observer (5)-(7). The red plot is the enlarged steady state estimation error.
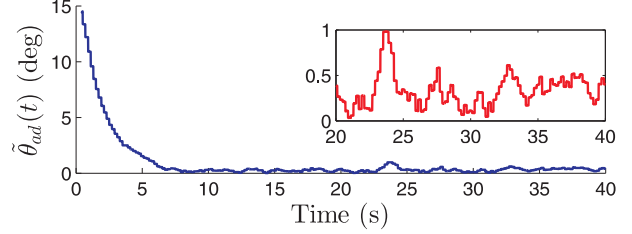

Fig. 5. Attitude estimation error of the ad-hoc observer. The red plot is the enlarged steady state estimation error.

innovation term is zero which simplifies the observer to a forward integration of attitude kinematics. This innovation term is mathematically formulated as follows.

$$\alpha(t) = \begin{cases} \bar{l}_1 z_1(t'_{k_1}) \times \hat{R}_{ad}(t'_{k_1} - \tau_1)^\top \mathring{y}_1, & t = t'_{k_1} \neq t'_{k_2} \\ \bar{l}_2 z_2(t'_{k_2}) \times \hat{R}_{ad}(t'_{k_2} - \tau_2)^\top \mathring{y}_2, & t = t'_{k_2} \neq t'_{k_1} \\ \sum_{i=1}^2 \bar{l}_i z_i(t'_{k_i}) \times \hat{R}_{ad}(t'_{k_i} - \tau_i)^\top \mathring{y}_i, & t = t'_{k_1} = t'_{k_2} \\ 0 & t \neq t'_{k_i} \end{cases}$$

This ad-hoc method adaptation of observers is commonly used in engineering applications to handle sensor sampling and delay effects (see e.g. [27], [28] for an EKF example).

The initial conditions of the combined predictor-observer (i.e. $\hat{R}(0.4)$ and $\Delta(0)$) and the initial condition of the ad-hoc observer (i.e. $\hat{R}_{ad}(t)$, $t \in [0, 0.4]$) are set to the identity matrix. The attitude estimation error of the combined predictor-observer is illustrated in Fig. 4, where the observer gains are chosen as $l_1 = l_2 = 0.5$. In this figure, the error $\tilde{\theta}$ is the angle of rotation in the angle-axis representation of the attitude estimate error $\hat{R}(t)R(t)^\top$ and is given by $\tilde{\theta}(t) = \frac{180}{\pi} \arccos(1 - 0.5\mathrm{tr}(I - \hat{R}(t)R(t)^\top))$. Note that the observer trajectories are available after the first sample of the vector measurements have been provided by the attitude sensors. The red plot shows the steady state estimation error which illustrates the good performance of our proposed method even with high sensor delay, low sampling rate, and high noise. Fig. 5 shows the estimation error $\tilde{\theta}_{ad}(t) = \frac{180}{\pi} \arccos(1 - 0.5\mathrm{tr}(I - \hat{R}_{ad}(t)R(t)^\top))$ of the ad-hoc observer when its gains are chosen as $\bar{l}_1 = \bar{l}_2 = 42.5$ such that the error trajectory of this observer has approximately the same transient convergence rate as Fig. 4. Comparing Fig. 4 and Fig. 5, the steady state error of our predictor-observer is almost an order of magnitude less than the steady state error of the ad-hoc observer. Next, we increase the sensor delays to $\rho_1 = \rho_2 = 0.5$ (s) and $\sigma_1 = \sigma_2 = 1.5$ (s) yielding a total sensor delay of 2 (s). With the same gains
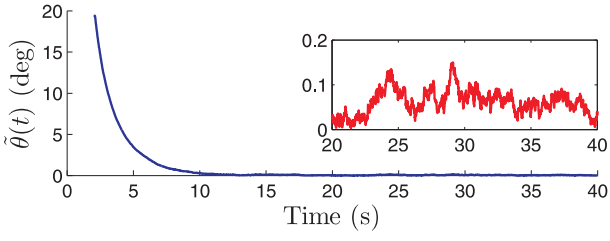
Fig. 6. Attitude estimation error of combined predictor-observer (5)-(7) with large sensor delay. The red plot is the enlarged steady state estimation error.

Fig. 8. Attitude estimation error of combined predictor-observer (5)-(7) with delay uncertainty. The small plots are the steady state estimation errors.
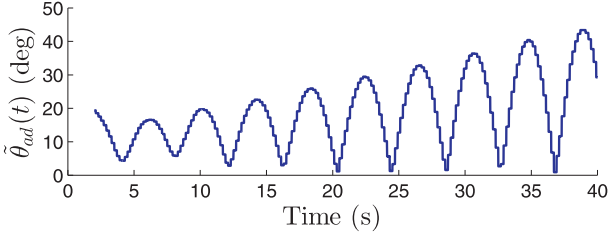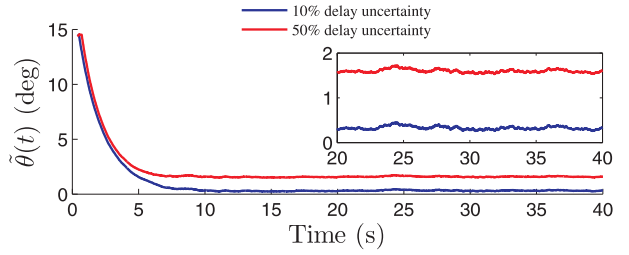

Fig. 7. Attitude estimation error of the ad-hoc observer with large sensor delay.

and initial conditions as in the previous simulation, the error trajectories of the predictor-observer and the ad-hoc observer are illustrated in Fig. 6 and Fig. 7, respectively. These plots show the convergence of the estimation error of our proposed predictor-observer while the estimation error of the ad-hoc observer diverges. The small degradation of the steady state estimation error of Fig. 6 compared to Fig. 4 is due to the fact that the predictor relies on noisy gyro measurements to compensate for the delay in vector measurements. Hence, a larger delay means longer integration of gyro noise which increases the estimation error. Nevertheless, the steady state estimation error of Fig. 6 is less than twice the corresponding error in Fig. 4 even though the sensor delay is increased by a factor of five.

Next, consider the same condition as the first simulation scenario, but, assume that there is uncertainty in knowledge of the amount of delay. To this end, we consider the sensor model of Fig. 1 with the same parameters as the first simulation but we consider two examples where the amount of the total delay that is used in the predictor (5)-(6) is either 10 or 50 percent more than the total delay in the simulated sensor model (i.e. $\tau_1 = \tau_2 = 0.44$ (s) or $\tau_1 = \tau_2 = 0.6$ (s), respectively). Fig. 8 shows that the estimation error is practically stable in both cases although the steady state estimation error is increased comparing to Fig 4. The steady state estimation errors are less than 0.5 (deg) and 1.8 (deg) respectively for 10% and 50% delay uncertainties which still demonstrate a very good performance considering the high values of noise and delay uncertainties.

V. CONCLUSION

We propose a combined predictor-observer methodology for the attitude estimation problem in the presence of sampled and delayed vector measurements. Exploiting the symmetries of the attitude kinematics and the system output maps, our proposed predictor is capable of reconstructing continuous-time delay free predictions of the vector measurements. The proposed predictor is generic and can be combined with arbitrary observers or filters. When combined with a geometric attitude observer, our proposed predictor-observer approach shows improved performance in simulation compared to Lyapunov-Krasovskii methods.

VI. ACKNOWLEDGMENT

APPENDIX

A. Equivalency of sensor models:

Setting $z_i(t) = w_i(t)$ for all t in (1) and (2) yields $y_i(t_{k_i}) = y_i(t'_{k_i} - \tau_i)$ and $[t_{k_i} + \sigma_i, t_{k_i+1} + \sigma_i) = [t'_{k_i}, t'_{k_i+1})$. These equalities hold if and only if $\tau_i = \rho_i + \sigma_i$ and $(t'_{k_i})_{k_i=1}^{\infty} = (t_{k_i} + \sigma_i)_{k_i=1}^{\infty}$. ■

B. Proof of Theorem 1:

The proof is based on application of the following Lemma that allows time-shifting of the attitude trajectory using the trajectory of the predictor.

Lemma 1: The trajectory $R(t)$ of the attitude kinematics (3) and the trajectory $\Delta(t)$ of the predictor dynamics (5) satisfy $R(t_2) = R(t_1)\Delta(t_1)^\top \Delta(t_2)$ for all $t_1, t_2 > 0$ and all $R_0, \Delta_0 \in SO(3)$. Moreover, assuming (4), the trajectory of $y_i(t)$ satisfies $y_i(t_2) = \Delta(t_2)^\top \Delta(t_1) y_i(t_1)$ for all $t_1, t_2 > 0$ and all $R_0, \Delta_0 \in SO(3)$. □

Proof of Lemma 1: Using (3) and (5) we have $\frac{d}{dt}(R(t)\Delta(t)^\top) = R(t)\Omega(t)_\times \Delta(t)^\top + R(t)\Omega(t)_\times^\top \Delta(t)^\top = 0$ for all $t \geq 0$. This means that $R(t)\Delta(t)^\top$ is constant for all $t \geq 0$ which in particular implies $R(t_1)\Delta(t_1)^\top = R(t_2)\Delta(t_2)^\top$ for all $t_1, t_2 \geq 0$ and proves the first claim of Lemma 1. Using (6) we have $y_i(t_2) = R(t_2)^\top \mathring{y}_i = \Delta(t_2)^\top \Delta(t_1) R(t_1)^\top \mathring{y}_i = \Delta(t_2)^\top \Delta(t_1) y_i(t_1)$ for all $t_1, t_2 \geq t_{1_i}$. This completes the proof. ■

Choosing $t_1 = t'_{k_i} - \tau_i$ and $t_2 = t$ and invoking Lemma 1 we have $y_i(t) = \Delta(t)^\top \Delta(t'_{k_i} - \tau_i) y_i(t'_{k_i} - \tau_i)$. Now, choosing $t \in [t'_{k_i}, t'_{k_i+1})$ and recalling (2) we have $y_i(t) = \Delta(t)^\top \Delta(t'_{k_i} - \tau_i) z_i(t)$ for all $t \in [t'_{k_i}, t'_{k_i+1})$ which together with (6) implies $y_i^p(t) = y_i(t)$ for all $t \geq t_{1_i}$. This completes the proof of Theorem 1.

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