Velocity Aided Attitude Estimation on SO(3) with Sensor Delay

Alireza Khosravian, Jochen Trumpf, Robert Mahony, Tarek Hamel

***Abstract*— This paper provides a nonlinear attitude estimation method for vehicles performing high acceleration maneuvers when measurements of linear velocity and the ambient magnetic field are delayed. Linear velocity is measured using the Global Positioning System (GPS) and the delays of GPS and magnetometer measurements are assumed to be known constants. Our proposed method consists of a delayed observer coupled with a dynamic predictor. The delayed observer uses delayed measurements and provides estimates of delayed attitude and velocity states that are in turn used in the predictor to generate estimates of the current attitude and velocity. A key contribution of the paper is effective use of the underlying symmetries of the system in order to design a generic predictor that can be coupled to different observers. We prove exponential convergence of the predicted attitude and velocity for a specific observer choice drawn from the literature.**

I. Introduction

The orientation of a vehicle with respect to a known reference frame is called its attitude. The attitude estimation problem has been the subject of a large body of research during the past decades (see [2] and the references therein). Interest in this research topic is continuing to be driven by the growing field of unmanned vehicles applications and the requirement for low-cost sensors along with robust and accurate low-computation algorithms.

The Kalman filter and its variants are the most popular attitude estimation methods [3], [10], [17]. While they exhibit good performance when properly designed and tuned, extended Kalman filters suffer from difficulty in tuning numerous parameters, high computational load, and unclear convergence properties in the nonlinear case [12].

Various representations of attitude are available, each of which has its own advantages [18]. The representation of attitude as an element of the special orthogonal group $\mathrm{SO}(3)$ is globally well defined, avoiding singularities that arise in Euler angle representations. The Lie-group representation of $\mathrm{SO}(3)$ encodes the complexity of the nonlinear attitude kinematics into the structure of the underlying group representation. This enables effective use of symmetries of the attitude kinematics system in order to design deterministic attitude observers that ensure (almost global) asymptotic stability of the estimation error and demonstrate excellent performance in practice [4], [7], [8], [11], [15], [19], [21], [22]. These observers rely on fusing two sets of sensor information to obtain a robust estimate of attitude. The first set is low frequency information obtained from *vectorial measurements* of known directions in the inertial frame. The second set is high frequency information obtained from measurements of angular velocity.

The typical sensors employed in practice to provide the required vectorial measurements are magnetometers and accelerometers. Employing only magnetometers is not enough to uniquely determine the attitude when the magnetic field is constant in the inertial frame [7], [12], [20]. When the vehicle performs low-acceleration maneuvers, the accelerometer output has been used as an approximate measurement of the direction of gravity that yields an additional vectorial measurement to uniquely estimate the attitude [5], [11], [21]. Although these schemes have a demonstrated track record in many applications, the underlying assumptions are not formally correct [13] and the algorithms fail when the vehicle is subjected to significant acceleration [6]. One way of addressing this problem is to use measurements of the linear velocity of the vehicle, obtained from GPS, and to modify the attitude observer to take into account the effect of the vehicle's acceleration on the actual output of the accelerometer. This is the core idea behind velocity aided attitude estimation methods [1], [6], [12], [16], [21]. These methods ensure (almost global) asymptotic or even exponential stability of the attitude estimation error during high acceleration maneuvers, are computationally cheap, and demonstrate good performance in practice [6], [12].

Velocity measurements provided by commercial GPS units are usually delayed with respect to the actual velocity of the vehicle. The delay can be several hundred milliseconds (and even up to half a second) long due to various environmental effects and in-sensor processing delays [9]. Low cost commercial magnetometers may also exhibit measurement delays in the order of tens of milliseconds [14]. These delays, if not compensated for properly, can significantly degrade the performance of attitude estimators and even lead to instability issues due to the nonlinear nature of attitude kinematics. This is particularly true of GPS delays. Attitude estimation with GPS measurement delay has been considered in [9] using a modified extended Kalman filtering approach. Despite its good performance in practice, this algorithm suffers from the common drawbacks associated with Kalman filters mentioned before. In particular, the required computational load is very high in [9] due to the propagation stages associated with delay compensation. The authors are unaware of any work prior to the present contribution that considers deterministic attitude observer design with sensor delays.

In this paper, we assume that the measurements of linear

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velocity and the magnetic field are delayed and we propose a method to estimate both the current attitude and the current linear velocity. The amount of delay in velocity and magnetometer readings is assumed to be constant and known. Our proposed method employs an observer that uses the delayed measurements and estimates the delayed attitude and velocity of the vehicle. We propose a dynamic predictor that uses the delayed estimates from the observer together with the current measurements of gyroscope and accelerometer in order to predict the current attitude and linear velocity of the vehicle (see Fig. 1). A key contribution of the paper is effective use of the underlying symmetries of the system in order to design the predictor such that the proposed observer-predictor pair guarantees exponential convergence of the predicted attitude and velocity to their current actual values. The proposed predictor is computationally cheap and it is generic in the sense that it can be combined with any asymptotic (exponential) attitude observer and still guarantees asymptotic (exponential) convergence of predicted attitude and velocity to their actual values. Numerical simulations are provided in the paper, showing the advantage of the proposed observer-predictor over the pure observers available in the literature in the presence of sensor delays. Via simulation, we also demonstrate the robustness of our proposed method against various sources of uncertainty, including sensor noise and bias.

The structure of the paper is as follows. Background and problem formulation is given in section II. The proposed observer-predictor approach is described in section III and its stability is proved by Theorem 1. Performance of our method is demonstrated via simulations in section IV.

II. Problem formulation

Motion of a rigid body moving in the Earth's gravitational field satisfies the following equations [6], [12], [16].

$$\dot{R}(t) = R(t)\Omega(t)_\times, \qquad R(0) = R_0 \tag{1}$$

$$\dot{v}(t) = g\mathrm{e}_3 + R(t)a(t), \quad v(0) = v_0 \tag{2}$$

where the attitude matrix $R \in \mathrm{SO}(3)$ is the rotation matrix describing the rotation of the body-fixed frame $\mathcal{B}$ with respect to the inertial frame $\mathcal{I}$, $\Omega \in \mathbb{R}^3$ is the angular velocity vector of $\mathcal{B}$ with respect to $\mathcal{I}$ expressed in $\mathcal{B}$, $v \in \mathbb{R}^3$ is the linear velocity of $\mathcal{B}$ with respect to $\mathcal{I}$ expressed in $\mathcal{I}$, $a \in \mathbb{R}^3$ is the so-called specific acceleration of the rigid body which represents the sum of all non-gravitational forces applied to the body divided by its mass and is expressed in $\mathcal{B}$, and $g\mathrm{e}_3$ is the (constant) gravitational acceleration vector expressed in $\mathcal{I}$. The linear operator $(.)_\times$ maps any vector in $\mathbb{R}^3$ to its corresponding skew-symmetric matrix in $\mathfrak{so}(3)$ such that $(x)_\times y$ is equal to the cross product $x \times y$ for all $x, y \in \mathbb{R}^3$. The internal states of the dynamical system (1)-(2) are the attitude matrix $R(t)$ and the velocity vector $v(t)$ and its inputs are the angular velocity vector $\Omega(t)$ and the specific acceleration $a(t)$.

We assume that the following sensors are attached to the rigid-body frame.

- A 3-axis gyro: measures the angular velocity $\Omega(t)$.
- A 3-axis accelerometer: measures the specific acceleration $a(t)$.
- A GPS unit: measures the linear velocity $v(t)$. Here, we assume that the GPS velocity measurement is available with a constant known delay τ_v. Denoting the GPS velocity measurement by $v_{\tau_v}(t)$ we have $v_{\tau_v}(t) = v(t - \tau_v)$.
- A 3-axis magnetometer: measures the magnetic field of the earth in the body-fixed frame. The magnetometer output $m(t)$ is related to the attitude matrix via $m(t) = R(t)^\top \mathring{m}(t)$ where $\mathring{m}(t)$ is the vector of the Earth's magnetic field at the position of the rigid body expressed in $\mathcal{I}$ and is assumed known for all $t > 0$. Here, we assume that the magnetometer measurement is available with a constant known delay τ_m. Denoting this delayed measurement by $m_{\tau_m}(t)$ we have $m_{\tau_m}(t) = R(t - \tau_m)^\top \mathring{m}(t - \tau_m)$.

The problem considered is to employ the measurements $\Omega(t)$, $a(t)$ and the delayed measurements $v_{\tau_v}(t)$ and $m_{\tau_m}(t)$ in order to obtain estimates of the current attitude matrix $R(t)$ and the current linear velocity $v(t)$. This particular problem setup is motivated by the typical scenario encountered in small scale unmanned aerial vehicle systems where the measurement delays of gyros and accelerometers can be ignored since they are very small compared to the GPS and magnetometer delays. We ignore measurement noise here to reduce the complexity at the design stage. We do, however, consider the effect of noise in the simulation section IV.

Notations and definitions:

Throughout the paper, $|.|$ denotes the Euclidean norm of a vector in $\mathbb{R}^3$ and $\|.\|$ denotes the Frobenius norm of a matrix in $\mathbb{R}^{3\times 3}$. We say $a(t) \in \mathbb{R}^3$ converges exponentially to $b(t) \in \mathbb{R}^3$ (resp. $A(t) \in \mathbb{R}^{3\times 3}$ converges exponentially to $B(t) \in \mathbb{R}^{3\times 3}$) if there exist positive constants γ and λ and a time t_0 such that $|a(t) - b(t)| \leq \gamma \exp(-\lambda(t - t_0))$ (resp. $\|A(t) - B(t)\| \leq \gamma \exp(-\lambda(t - t_0))$) for all $t \geq t_0$.

III. Observer-predictor approach

Defining $\tau := \max(\tau_v, \tau_m)$, the values of $v(t - \tau)$ and $m(t - \tau)$ are available at time t by time-shifting the magnetometer or the GPS output. The approach that we take here to tackle the problem described in the previous section is illustrated in Fig. 1. First, we employ an observer that uses the shifted measurements to obtain an estimate of the delayed state $(R(t - \tau), v(t - \tau))$. Then we propose a predictor that uses the delayed estimate of the state together with the inputs of the system (i.e. $\Omega(t)$ and $a(t)$) and predicts the current state $(R(t), v(t))$.

Denoting the estimates of $R(t - \tau)$ and $v(t - \tau)$, respectively, by $\hat{R}_\tau(t) \in \mathrm{SO}(3)$ and $\hat{v}_\tau(t) \in \mathbb{R}^3$, we time-shift the observer proposed in [6, equation (6)] for the case of

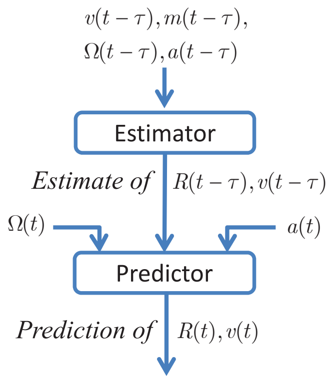

Fig. 1. Illustration of the proposed observer-predictor (3)-(9)

delay-free measurements to obtain the following observer.

$$\dot{\hat{R}}_\tau(t) = \hat{R}_\tau(t)\left(\Omega(t-\tau)+\sigma\right)_\times, \tag{3}$$

$$\begin{aligned}\sigma = {} & k_1 m(t-\tau) \times \hat{R}_\tau(t)^\top \mathring{m}(t-\tau) \\ & + k_2 a(t-\tau) \times \hat{R}_\tau(t)^\top (v(t-\tau) - \hat{v}_\tau(t)),\end{aligned} \tag{4}$$

$$\dot{\hat{v}}_\tau(t) = g e_3 + \hat{R}_\tau(t) a(t-\tau) + k_3(v(t-\tau) - \hat{v}_\tau(t)) \tag{5}$$

for $t \geq \tau$ with the initial values $\hat{R}_\tau(\tau) = \hat{R}_{\tau_0}$ and $\hat{v}_\tau(\tau) = \hat{v}_{\tau_0}$, and constant positive gains k_1, k_2, and k_3. Note that $\Omega(t-\tau)$, $a(t-\tau)$, $v(t-\tau)$, and $m(t-\tau)$ are all available at time t.

We combine the observer (3)-(5) with the following predictor.

$$\dot{\Delta}(t) = \Delta(t)\Omega_\times(t), \qquad t \geq 0 \tag{6}$$

$$\hat{R}(t) = \hat{R}_\tau(t)\Delta(t-\tau)^\top\Delta(t), \qquad t \geq \tau \tag{7}$$

$$\dot{\delta}(t) = g e_3 + \hat{R}(t)a(t), \qquad t \geq \tau \tag{8}$$

$$\hat{v}(t) = \hat{v}_\tau(t) + \delta(t) - \delta(t-\tau), \qquad t \geq 2\tau \tag{9}$$

with the initial conditions $\Delta(0) = \Delta_0 \in \mathrm{SO}(3)$ and $\delta(\tau) = \delta_0 \in \mathbb{R}^3$. The variables $\Delta(t) \in \mathrm{SO}(3)$ and $\delta(t) \in \mathbb{R}^3$ are internal states of the predictor and the variables $\hat{R}(t)$ and $\hat{v}(t)$ are predictions of $R(t)$ and $v(t)$, respectively. The actual role of the internal predictor dynamics (6) and (8) is to use the information contained in the inputs $\Omega(t)$ and $a(t)$ from time $t-\tau$ to time t to predict the *increment* of attitude and linear velocity during that time period. These predictions of the increments of the states are given by the term $\Delta(t-\tau)^\top\Delta(t)$ in (7) and the term $\delta(t) - \delta(t-\tau)$ in (8).

We define the following errors.

$$\tilde{R}_\tau(t) = \hat{R}_\tau(t)R(t-\tau)^\top, \qquad t \geq \tau \tag{10}$$

$$\tilde{v}_\tau(t) = \hat{v}_\tau(t) - v(t-\tau), \qquad t \geq \tau \tag{11}$$

$$\tilde{R}(t) = \hat{R}(t)R(t)^\top, \qquad t \geq \tau \tag{12}$$

$$\tilde{v}(t) = \hat{v}(t) - v(t) \qquad t \geq 2\tau \tag{13}$$

$\tilde{R}_\tau$ and $\tilde{v}_\tau$ evaluate the estimation error of the observer (3)-(5) while $\tilde{R}$ and $\tilde{v}$ evaluate the total error of the observer-predictor combination (3)-(9).

We require the following mild assumptions to derive the convergence properties of the proposed observer-predictor.

Assumption 1: There exist positive constants $\underline{c}_a$, $\overline{c}_a$, c_Ω, c_v, $c_{\ddot{v}}$ such that for all $t \geq 0$ we have $\underline{c}_a \leq |a(t)| \leq \overline{c}_a$, $|\Omega(t)| \leq c_\Omega$, $|v(t)| \leq c_v$, and $|\ddot{v}(t)| \leq c_{\ddot{v}}$.

Assumption 2 (Observability condition): There exists a positive constant c_{obs} such that for all $t > 0$ we have $|\mathring{m}(t) \times (\dot{v}(t) - g e_3)| \geq c_{\text{obs}}$.

Assumptions 1 and 2 are similar to those imposed in [6] and are required to ensure the stability of estimation errors (10)-(11). The following Theorem summarizes the properties of the proposed observer-predictor.

Theorem 1: Consider the system (1)-(2), with the observer (3)-(5) and the predictor (6)-(9). Suppose that assumptions 1 and 2 are satisfied. Then

(a) for any values of the gains $k_1 > 0$, $k_2 > 0$ and $k_3 > 0$, the dynamics of the estimation error $(\tilde{R}_\tau(t), \tilde{v}_\tau(t))$ is locally exponentially stable at $(I, 0)$. Moreover, the total error $(\tilde{R}(t), \tilde{v}(t))$ converges exponentially to $(I, 0)$ for all initial values (Δ_0, δ_0) and $(\hat{R}_\tau(\tau), \hat{v}_\tau(\tau))$ that ensure the exponential stability of $(\tilde{R}_\tau(t), \tilde{v}_\tau(t))$.

(b) (Basin of attraction) for any closed neighborhood $\mathcal{N}$ of $(I, 0)$ with $\mathcal{N} \subset \{(R, v) \in \mathrm{SO}(3) \times \mathbb{R}^3 : \ \mathrm{tr}(R) \neq -1\}$, there exists a constant $K > 0$ such that for all $k_3 > K$, all $k_1 > 0$ and $k_2 > 0$, and all initial errors $(\tilde{R}_\tau(\tau), \tilde{v}_\tau(\tau)) \in \mathcal{N}$, the estimation error $(\tilde{R}_\tau(t), \tilde{v}_\tau(t))$ converges exponentially to $(I, 0)$. Moreover, the total error $(\tilde{R}(t), \tilde{v}(t))$ converges exponentially to $(I, 0)$ for all initial values of (Δ_0, δ_0) and all $(\tilde{R}_\tau(\tau), \tilde{v}_\tau(\tau)) \in \mathcal{N}$.

(c) For both parts (a) and (b), $\delta(t)$ is bounded for all $t \geq \tau$.

□

Proof of Theorem 1 is given in the appendix.

Under ideal conditions, boundedness of $\delta(t)$ is guaranteed by part (c) of Theorem 1. Nevertheless, in practical situations $\delta(t)$ can potentially grow unbounded due to sensor noise or inaccuracy of the numerical integration procedure used to implement (8). One way of addressing this problem is to add a Lyapunov-Krasovskii feedback term to (8) to stabilize the dynamics of δ. However, this significantly complicates the proof of convergence of the total error. An alternative method is to employ two copies of (8) and use a simple switching technique which bounds the trajectory of their internal states while maintaining the convergence of the total error.

Lemma 1: Consider deterministic noise $n(t)$ that models the measurement noise and the inaccuracy of numerical integration. Assume that positive constants $\overline{n}$ and $\overline{c}_a$ exist such that $|n(t)| \leq \overline{n}$ and $|a(t)| \leq \overline{c}_a$ for all $t \geq 0$. Consider the following systems.

$$\dot{\delta}_1(t) = g e_3 + \hat{R}(t)a(t) + n(t), \qquad t \geq \tau \tag{14}$$

$$\dot{\delta}_2(t) = g e_3 + \hat{R}(t)a(t) + n(t), \qquad t \geq 2\tau \tag{15}$$

where the initial condition of (14) is reset to the value δ_0 at times $t = (2i-1)\tau$, $i = 1, 2, \ldots$ and the initial condition of (15) is reset to δ_0 at times $t = 2i\tau$. Define the following trajectory for $t \geq 2\tau$ and $i = 1, 2, \ldots$.

$$\overline{\delta}(t) = \begin{cases} \delta_1(t) - \delta_1(t-\tau), & 2i\tau \leq t < (2i+1)\tau \\ \delta_2(t) - \delta_2(t-\tau), & (2i+1)\tau \leq t < 2(i+1)\tau \end{cases}$$

Then the following statements hold true.

(a) If $n(t) = 0$ then we have $\bar{\delta}(t) = \delta(t) - \delta(t-\tau)$ for all $t \geq \tau$.

(b) The trajectories of $\delta_1(t)$ (for $t \geq \tau$) and $\delta_2(t)$ (for $t \geq 2\tau$) are bounded. Moreover, the error $|\bar{\delta}(t) - (\delta(t) - \delta(t-\tau))|$ is bounded by $\tau\bar{n}$ for all $t \geq \tau$.

□

Proof of Lemma 1 is given in the appendix. Using this Lemma, the dynamics (14)-(15) can be employed instead of (8) and the predictor (9) can be replaced by $\hat{v}(t) = v(t-\tau) + \bar{\delta}(t)$. The advantage of using Lemma 1 is that $\delta_1(t)$ and $\delta_2(t)$ remain bounded for all times due to the resetting of the initial conditions of (14)-(15) every 2τ seconds.

Remark 1: When delay-free measurements are available, several observers for velocity aided attitude estimation have been proposed in the literature each of which has its own advantages [6], [12], [16]. We opt to employ the observer proposed in [6, equation (6)] since it ensures exponential stability of the estimation error $(\tilde{R}_\tau(t), \tilde{v}_\tau(t))$ which in turn simplifies the proof of convergence for the total error $(\tilde{R}(t), \tilde{v}(t))$. Nevertheless, the proposed predictor (6)-(9) is generic in the sense that it can be combined with any other observer as well. In this case, the total error $(\tilde{R}(t), \tilde{v}(t))$ converges asymptotically (exponentially) to $(I, 0)$ if the estimation error $(\tilde{R}_\tau(t), \tilde{v}_\tau(t))$ is asymptotically (exponentially) stable at $(I, 0)$ (this is a consequence of Lemma 2 in the appendix). If the stability of the employed observer is only asymptotic, then $\delta(t)$ is bounded if there exists a constant c such that $\int_\tau^t \|I - \tilde{R}_\tau(s)\|^2 ds \leq c$ for all $t \geq \tau$, i.e. if the attitude observer error converges with finite energy. In any case, we can bound the trajectory of $\delta(t)$ using the basic switching technique explained in Lemma 1. The proposed predictor (6)-(7) can also be combined with the pure attitude observer of [4], [7], [11], [15], [20]–[22] to encounter delays in the vector measurements or their corresponding reference vectors. □

Remark 2: In practice, usually the GPS delay τ_v is much larger than the magnetometer delay τ_m and hence $\tau = \tau_v$. In this case, the construction of the observer (3)-(5) is such that it does not use the most current magnetometer measurement $m_{\tau_m}(t)$ but instead it only uses the delayed magnetometer measurement $m_{\tau_m}(t - \tau_v + \tau_m) = m(t-\tau)$. It is possible to extend the method presented in this paper to employ the most current magnetometer measurement as well. To this end, one can consider multiple copies of the observer-predictor approach presented here in a serial cascade combination such that each observer-predictor pair uses the measurements at $t - \tau_1$ and provides the prediction of the state at $t - \tau_2$ with $\tau_2 < \tau_1$. More precisely, the approach that we can take is to combine the observer (3)-(5) with a predictor that takes the estimate of $(R(t-\tau_v), v(t-\tau_v))$ from the observer and predicts the delayed states $(R(t-\tau_m), v(t-\tau_m))$. Then we can use the prediction of $(R(t-\tau_m), v(t-\tau_m))$ together with the current measurement of the magnetometer $m_{\tau_m}(t) = m(t-\tau_m)$ in a time-shifted observer that estimates the delayed states $(R(t-\tau_m), v(t-\tau_m))$. Finally, we can use the estimate of $(R(t-\tau_m), v(t-\tau_m))$ in a predictor to obtain a prediction of the current state $(R(t), v(t))$. In practice, if there is no constraint in computational load, the decision about whether or not to use the multi-stage observer-predictor comes down to the accuracy of the employed magnetometer compared to the accuracy of the employed gyro and accelerometer. If the accuracy of the measurement of the magnetic field is high, and its delay is significantly smaller than the GPS delay, then it is worth using the multi-stage observer-predictor to include the extra measurement $m(t-\tau_m)$ in estimating the current states. Otherwise, it is better to employ the simple observer-predictor (3)-(9). □

IV. Simulation results

In this section, we illustrate the performance of the proposed observer-predictor. We provide two sets of simulations. In simulation 1, we compare the performance of the observer-predictor (3)-(9) with the pure observer proposed in [6]. In simulation 2, we investigate the robustness of our proposed observer-predictor against various sources of noise that are encountered in practice.

To generate the trajectories of $R(t)$ and $v(t)$, we assumed that the vehicle performs circular manoeuvres in the horizontal plane such that its angular velocity is $\Omega = [0,\ 0,\ \omega_0]^\top$ with $\omega_0 = \frac{\pi}{4}$ (rad/s) and its linear velocity is $v(t) = [-5\omega_0 \sin(\omega_0 t),\ 5\omega_0 \cos(\omega_0 t),\ 0]^\top$ (m/s). The initial attitude of the vehicle corresponds to an orientation of roll 80 (deg), pitch 10 (deg), and yaw 0 (deg). The initial value of the observer (3)-(5) is set to $(\hat{R}_\tau(0), \hat{v}_\tau(0)) = (I, 0)$ corresponding to a large initial estimation error. The observer gains are chosen as $k_1 = 3$, $k_2 = 0.03$ and $k_3 = 3$. The normalized magnetic field is taken as $\mathring{m} = \frac{1}{\sqrt{2}}[1,\ 1,\ 0]^\top$ and the initial value of the predictor (6)-(9) is set to $(\Delta_0, \delta_0) = (I, 0)$.

Simulation 1:

We consider a velocity measurement delay of $\tau_v = 1$ (s) and zero delay for the magnetometer, i.e. $\tau = 1$ (sec). The chosen amount of velocity measurement delay ensures that the direction of the delayed velocity $v(t-\tau_v)$ is always 45 (deg) different from that of the current velocity $v(t)$. We feed the observer of [6, equation (6)] with the delayed velocity $v(t-\tau_v)$ rather than with the current velocity $v(t)$. The initial values and gains of this observer are set to the same values as for our delayed observer (3)-(5). Fig. 2 shows that the attitude and velocity estimation errors of the observer [6, equation (6)] do not converge to zero when it is fed with delayed velocity measurements. Fig. 3 and Fig. 4 show the estimation errors $\|I - \hat{R}_\tau(t)R(t-\tau)^\top\|$ and $|\hat{v}_\tau(t) - v(t-\tau)|$ of the delayed observer (3)-(5) and the total errors $\|I - \hat{R}(t)R(t)^\top\|$ and $|\hat{v}(t) - v(t)|$ of the observer-predictor (3)-(9), respectively. Observe that the delayed observer (3)-(5) starts working at time $t = \tau$, hence its estimate is available for $t \geq \tau$. Similarly, the prediction of the attitude $\hat{R}(t)$ and the velocity $\hat{v}(t)$ from (6)-(9) are available for $t \geq \tau$ and $t \geq 2\tau$, respectively. Fig. 3 and Fig. 4 show the convergence of both estimation errors and both total errors to zero. It is also clear by comparing the first plots of Fig. 3 and Fig. 4

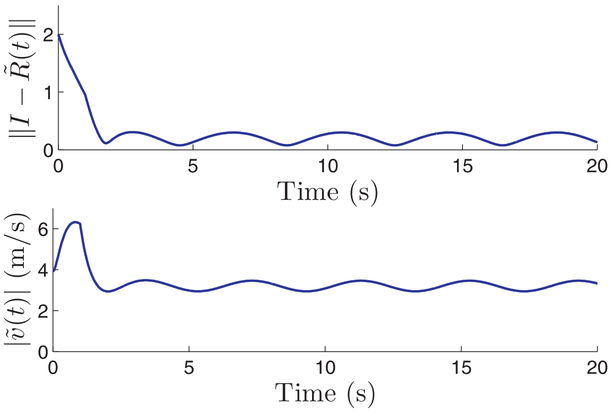

Fig. 2. Attitude estimation error and velocity estimation error for the observer [6, equation (6)] when fed with the delayed measurement $v(t-\tau)$.

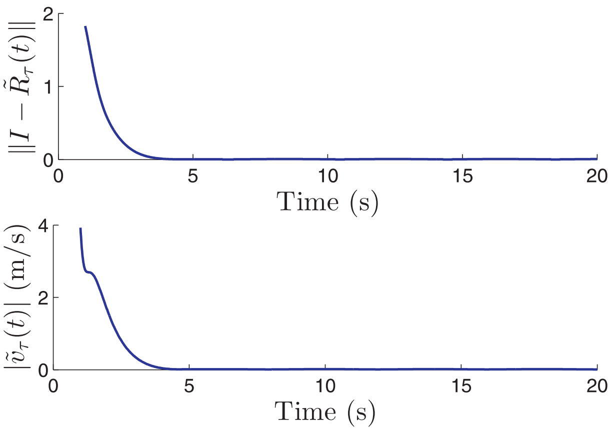

Fig. 3. Attitude estimation error and velocity estimation error for the observer (3)-(5).

that the total attitude error $\|I - \tilde{R}(t)\|$ equals the delayed attitude estimation error $\|I - \tilde{R}_\tau(t)\|$ as shown in Lemma 2 in the appendix. Finally, Fig. 5 shows that the internal state $\delta(t)$ of the predictor is bounded for all time, as shown in part (c) of Theorem 1.

Simulation 2:

In this simulation, various sources of noise are considered to investigate the robustness of the proposed observer-predictor. We disturb each axis of measurement of the angular velocity $\Omega(t)$, the linear velocity $v(t)$, the acceleration $a(t)$, and the magnetometer output $m(t)$ by zero-mean additive Gaussian noise processes with standard deviations of 1 (deg/s), 0.1 (m/sec), 0.1 (m/s^2), and 0.1, respectively. In addition, we consider a constant bias of $[1, \ -1, \ 1]^\top$ (deg/s) for the gyro and a bias of $[0.1, \ -0.1, \ 0.1]^\top$ (m/s^2) for the accelerometer. Moreover, we consider $\bar{\tau} = 1.2$ (s) instead of the actual value $\tau = 1$ (s) in the implementation of the observer-predictor (3)-(9) while the measurement of linear velocity that is fed to the observer-predictor is still delayed by $\tau = 1$ (s). The difference between $\bar{\tau}$ and τ models the uncertainty in knowledge of the real sensor delay. The switching method described in Lemma 1 is employed to implement the predictor. Under these conditions, the

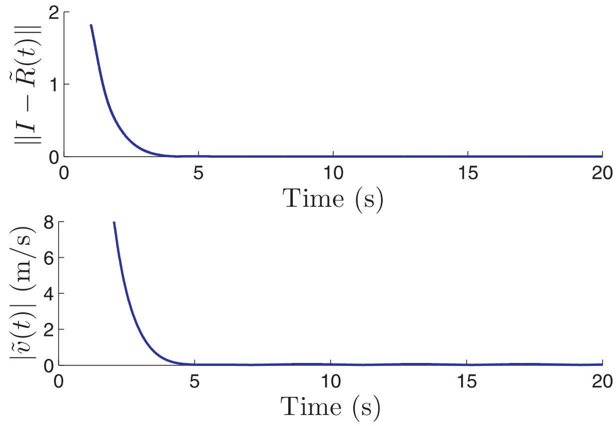

Fig. 4. Total attitude error and total velocity error for the observer-predictor combination (3)-(9).

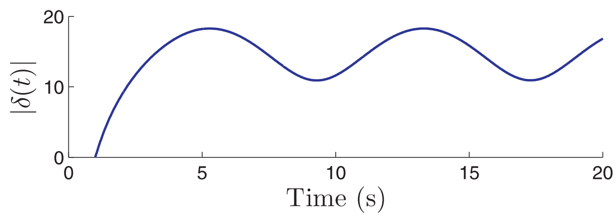

Fig. 5. Internal state $\delta(t)$ of the predictor (8)-(9).

estimation error of the observer (3)-(5) and the total error of the observer-predictor (3)-(9) are illustrated in Fig. 6 and Fig. 7, respectively. These plots show the good performance of the proposed observer-predictor despite the high amount of noise, the presence of bias, and the uncertainty in the delay. The steady state error trajectories have been enlarged in the figures, showing that the maximum absolute attitude estimation error of the observer (3)-(5) is less than 0.9 (equivalent to a rotation of 3.7 (deg)) and the maximum absolute total attitude error of the observer-predictor (3)-(9) is less than 0.12 (equivalent to a rotation of 4.9 (deg)). The steady-state maximum absolute velocity estimation error and total velocity error are respectively about 0.75 (m/s) and 1.4

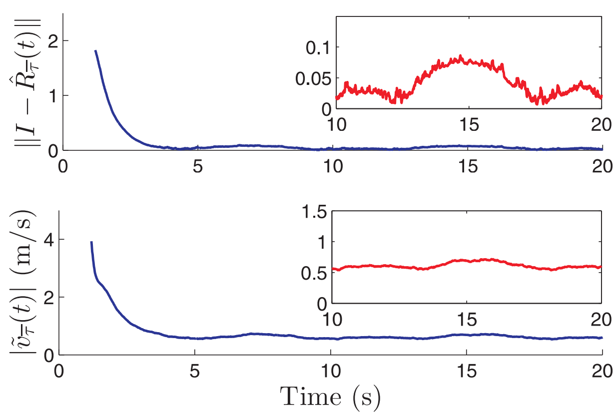

Fig. 6. Attitude estimation error $\|I - \tilde{R}_{\bar{\tau}}(t)\| = \|I - \hat{R}_\tau(t)R(t-\bar{\tau})^\top\|$ and velocity estimation error $|\tilde{v}_{\bar{\tau}}(t)| = |\hat{v}_\tau(t) - v(t-\bar{\tau})|$ for the observer (3)-(5) (non-ideal measurements). Red plots are steady state errors.

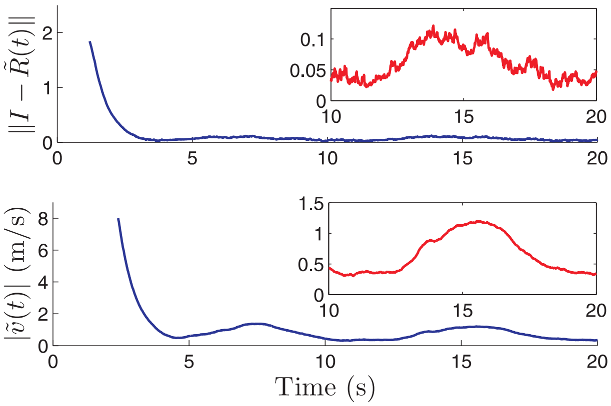

Fig. 7. Total attitude error and total velocity error for the observer-predictor combination (3)-(9) (non-ideal measurements). Red plots are steady state errors.

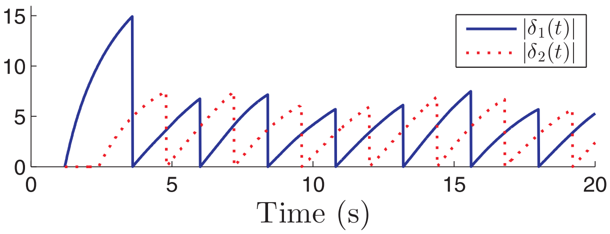

Fig. 8. Amplitude of the internal states of the predictor when it is implemented using the switching dynamics (14)-(15) (non-ideal measurements).

(m/s). This is still good considering that the chosen trajectory of the vehicle is such that $|v(t)-v(t-\tau)|\approx 3$ (m/s) for all $t\geq\tau$. A bias in the velocity estimation error is clearly visible in Fig. 6 and is largely due to the modelled sensor bias and the uncertainty in the knowledge of τ.

V. Conclusions

We propose a coupled observer-predictor to obtain estimates of the attitude and velocity of a vehicle when the magnetometer and linear velocity measurements (the latter provided by GPS) are delayed by a constant known amount. We prove that the estimates of the current attitude and velocity converge to their actual values exponentially fast. A key contribution of the paper is effective use of the underlying symmetries of the system in order to design a generic predictor that can be combined with any asymptotically stable observer, as well as the one employed here, and still guarantees convergence of the predicted attitude and velocity to their actual values. In future work, estimator design with time-varying or unknown delays, and also adaptive compensation of sensor bias can be considered.

Appendix

Proof of Theorem 1:

The observer (3)-(5) has exactly the same form as the observer proposed in [6, equation (6)] for the case of delay-free measurements. The only difference is that we have time-shifted the observer to time $t-\tau$. A direct application of [6, Theorem 1] guarantees that $(\tilde{R}_\tau(t),\tilde{v}_\tau(t))$ is locally exponentially stable at $(I,0)$ and its basin of attraction is given by $\mathcal{N}$ as defined in part (b). Proof of convergence of the total error $(\tilde{R}(t),\tilde{v}(t))$ to $(I,0)$ is based on the following Lemma.

Lemma 2: Consider the attitude kinematics (1) together with the observer (3)-(5) and the predictor (6)-(7). Then, the total error $\tilde{R}(t)$ defined by (12) equals the estimation error $\tilde{R}_\tau(t)$ defined by (10) for all $t\geq\tau$. □

Proof of Lemma 2: Consider the following system

$$\dot{\Delta}_I(t)=\Delta_I(t)\Omega(t)_\times,\qquad \Delta_I(0)=I \tag{16}$$

where $\Delta_I(t)\in\mathrm{SO}(3)$. System (16), the attitude kinematics (1), and the predictor's internal dynamics (6) are copies of the same left invariant system with the same input $\Omega(t)$ but with different initial conditions. It is easy to verify that the solution $R(t)$ of the attitude kinematics (1) and the solution $\Delta(t)$ of (6) are related to the solution $\Delta_I(t)$ of (16) via

$$R(t)=R_0\Delta_I(t), \tag{17}$$
$$\Delta(t)=\Delta_0\Delta_I(t) \tag{18}$$

for all $t\geq 0$ where R_0 and Δ_0 are the initial conditions of (1) and (6), receptively. Combining (17) and (18) yields $R(t)\Delta(t)^\top=R_0\Delta_0^\top$ for all $t\geq 0$. Since the right-hand side of this equation is constant, it follows that $R(t)\Delta(t)^\top=R(t-\tau)\Delta(t-\tau)^\top$ for all $t\geq\tau$ and hence

$$R(t)=R(t-\tau)\Delta(t-\tau)^\top\Delta(t) \tag{19}$$

for all $t\geq\tau$. Replacing $R(t)$ and $\hat{R}(t)$ from (19) and (7) into (12) implies $\tilde{R}(t)=\hat{R}(t-\tau)\Delta(t-\tau)^\top\Delta(t)\Delta(t)^\top\Delta(t-\tau)R(t-\tau)^\top=\hat{R}(t-\tau)R(t-\tau)^\top=\tilde{R}_\tau(t)$ for all $t\geq\tau$.■

Using the proof of Theorem 1 in [6], it is easy to show that for any initial condition $(\tilde{R}_\tau(\tau),\tilde{v}_\tau(\tau))\in\mathcal{N}$, there exist positive constants $\gamma_1,\gamma_2,\lambda_1$ and λ_2 such that

$$\|I-\tilde{R}_\tau(t)\|\leq\gamma_1\exp(-\lambda_1(t-\tau)), \tag{20}$$
$$\|\tilde{v}_\tau(t)\|\leq\gamma_2\exp(-\lambda_2(t-\tau)) \tag{21}$$

for all $t\geq\tau$ where $\gamma_1,\gamma_2,\lambda_1,\lambda_2$ depend on the initial observer error $(\tilde{R}_\tau(\tau),\tilde{v}_\tau(\tau))$ as well as on the observer gains and the constants defined in Assumptions 1 and 2. Direct application of Lemma 2 and (20) yields $\|I-\tilde{R}(t)\|\leq\gamma_1\exp(-\lambda_1(t-\tau))$ for all $t\geq\tau$. This yields exponential convergence of $\tilde{R}(t)$ to I. It remains to show the exponential convergence of $\tilde{v}(t)$ to zero and boundedness of $\delta(t)$. Integrating both sides of (2) resp. (8) from $t-\tau$ to t we have

$$v(t)-v(t-\tau)=\int_{t-\tau}^{t}(ge_3+R(s)a(s))ds, \tag{22}$$

$$\delta(t)-\delta(t-\tau)=\int_{t-\tau}^{t}(ge_3+\hat{R}(s)a(s))ds, \tag{23}$$

for all $t\geq\tau$. Using (9) and employing (22) and (23) yields $\hat{v}(t)-v(t)=\hat{v}_\tau(t)-v(t-\tau)+\int_{t-\tau}^{t}(\hat{R}(s)-R(s))a(s)ds$

for all $t \geq 2\tau$. This together with (21) implies

$$|\tilde{v}(t)| \leq |\hat{v}_\tau(t) - v(t-\tau)| + \int_{t-\tau}^{t} \|I - \tilde{R}(s)\||a(s)|ds$$
$$\leq \gamma_2 \exp(-\lambda_2(t-\tau)) + \gamma_1 \bar{c}_a \int_{t-\tau}^{t} \exp(-\lambda_1(s-\tau))ds$$
$$= \gamma_2 \exp(-\lambda_2(t-\tau)) + \frac{\gamma \bar{c}_a (1-\exp(-\lambda_1 \tau))}{\lambda_1} \exp(-\lambda_1(t-2\tau))$$

for all $t \geq 2\tau$. Hence $\tilde{v}(t)$ converges exponentially to zero.

Integrating both sides of (2) resp. (8) from τ to $t \geq \tau$ we derive $\delta(t) - v(t) = \delta(\tau) - v(\tau) + \int_\tau^t (\hat{R}(s) - R(s))a(s)ds$ and hence

$$|\delta(t) - v(t)| \leq |\delta(\tau) - v(\tau)| + \int_{\tau}^{t} \|I - \tilde{R}(s)\||a(s)|ds$$
$$\leq |\delta(\tau) - v(\tau)| + \gamma_1 \bar{c}_a \int_{\tau}^{t} \exp(-\lambda_1(t-\tau))ds$$
$$\leq |\delta(\tau) - v(\tau)| + \frac{\gamma_1 \bar{c}_a}{\lambda_1}(1 - \exp(-\lambda_1(t-\tau))$$

for all $t \geq \tau$. Hence, $|\delta(t) - v(t)|$ is bounded for all $t \geq \tau$. This ensures that $\delta(t)$ is bounded since $v(t)$ is bounded (by assumption 1). This completes the proof of Theorem 1. ■

Proof of Lemma 1:

Proof of part (a): Assuming $n = 0$ and noting that the initial condition of (14) is not reset in the period $(2i-1)\tau < t < (2i+1)\tau$, we can integrate both sides of (14) from $t-\tau$ to t to obtain $\bar{\delta}(t) = \delta_1(t) - \delta_1(t-\tau) = \int_{t-\tau}^t (g\mathrm{e}_3 + \hat{R}(s)a(s))ds$ for $2i\tau \leq t < (2i+1)\tau$. On the other hand, integrating both sides of (8) we have $\delta(t) - \delta(t-\tau) = \int_{t-\tau}^t (g\mathrm{e}_3 + \hat{R}(s)a(s))ds$ for all $t \geq \tau$. Hence $\bar{\delta}(t) = \delta(t) - \delta(t-\tau)$ for $2i\tau \leq t < (2i+1)\tau$. Similarly, we can integrate both sides of (15) to obtain $\bar{\delta}(t) = \delta(t) - \delta(t-\tau)$ for $(2i+1)\tau \leq t < 2(i+1)\tau$ and together this implies $\bar{\delta}(t) = \delta(t)$ for all $t \geq \tau$.

Proof of part (b): Integrating (14) from $(2i-1)\tau$ to t and noting that $\delta((2i-1)\tau) = \delta_0$ we have $\delta_1(t) = \delta_0 + \int_{(2i-1)\tau}^t (g\mathrm{e}_3 + \hat{R}(s)a(s) + n(s))ds = \delta_0 + g(t-(2i-1)\tau)\mathrm{e}_3 + \int_{(2i-1)\tau}^t (\hat{R}(s)a(s) + n(s))ds$ for $(2i-1)\tau \leq t < (2i+1)\tau$. Since $|\mathrm{e}_3| = 1$, $|a(t)| \leq \bar{c}_a$ and $|n(t)| \leq \bar{n}$ we have

$$|\delta_1(t)| \leq |\delta_0| + g \cdot 2\tau + |\int_{(2i-1)\tau}^{t} (\hat{R}(s)a(s) + n(s))ds|$$
$$\leq |\delta_0| + g \cdot 2\tau + \int_{(2i-1)\tau}^{t} (|\hat{R}(s)a(s)| + |n(s)|)ds$$
$$\leq |\delta_0| + 2\tau(g + \bar{c}_a + \bar{n})$$

for all $(2i-1)\tau \leq t < (2i+1)\tau$. Since this holds for all $i = 1, 2, \ldots$, we conclude that $\delta_1(t)$ is bounded for all $t \geq \tau$. The same argument applied to (15) shows that $\delta_2(t)$ is bounded for all $t \geq 2\tau$. It remains to show that $|\bar{\delta}(t) - (\delta(t) - \delta(t-\tau))|$ is bounded. For $2i\tau \leq t < (2i+1)$ we have

$$\bar{\delta}(t) = \delta_1(t) - \delta_1(t-\tau) = \int_{t-\tau}^{t} \hat{R}(s)a(s)ds + \int_{t-\tau}^{t} n(s)ds$$
$$= \delta(t) - \delta(t-\tau) + \int_{t-\tau}^{t} n(s)ds$$

and similarly for $(2i+1)\tau \leq t < 2(i+1)\tau$, using $\delta_2(t)$ in place of $\delta_1(t)$. The result follows. ■

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