Nonlinear attitude observers on $\mathrm{SO}(3)$ for complementary and compatible measurements: A theoretical study

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Abstract— **This paper considers the question of designing an attitude observer exploiting the structure of the Special Orthogonal Group $\mathrm{SO}(3)$ for both inertial and body-fixed-frame measurements. We consider measurements from a minimal sensor suite, typically a rate gyroscope along with several measurements of inertial and/or body-fixed vector directions. We propose fully nonlinear pose observers based directly on the vector measurements, allowing a mix of both inertial and body-fixed measurements. We provide a comprehensive stability analysis of the observer error dynamics. We show that even with a single vector measurement it is often possible to recover asymptotically stable observer error dynamics.**

I. INTRODUCTION

Attitude or orientation estimation is a key requirement for autonomous operation of robotic vehicles. The recent development of affordable and lightweight Micro-Electro-Mechanical Systems (MEMS) components has led to the development of a range of low cost and light weight inertial measurement units that can be used to provide reliable measurements of angular velocity, magnetic field orientation and an approximation of the direction of gravity. These systems can be augmented with other sensors such as Global Positioning Systems (GPS), visual systems, infrared or acoustic range sensors, etc, that are primarily position sensors, however, can also sometimes be used to provide orientation information. For example, carrier phase doppler shift can be extracted from GPS signals to provide inertial frame velocity estimates of a vehicle's motion [8], which in turn provides orientation information for the motion of a non-holonomic vehicle such as a car. The last five years has seen a considerable effort in the development of nonlinear observers for attitude estimation. An important development that came from early work on estimation and control of satellites was the use of the quaternion representation for observer design [13], [6], [18], [15], [14]. The non-linear observer designs that are based on this work have strong robustness properties, however, they are limited by the requirement of a full orientation state measurement [5]. A general framework for observer analysis and design for systems with symmetry invariance was proposed in [1] and has led to a number of observer designs for attitude estimation [10], [3], [2]. This approach has been developed to provide a coherent design methodology for the design of attitude heading reference systems (AHRS) [11], [12]. The limitation of nonlinear observers in requiring full state measurements was first overcome in an ad-hoc manner in [7] with more insight provided in [9]. Closely related observers were developed independently in [4], [17]. Modern nonlinear attitude observers are simple to design, can be implemented on low-power embedded architectures, take partial attitude measurements as inputs directly, and are associated with a global robust stability analysis [9], [2]. The published results mostly consider the more common situation where the attitude measurement is made relative to the body-fixed-frame, for example, an estimate of the gravitational or magnetic field direction obtained from an inertial measurement unit (IMU). In the case where only a single body-fixed-frame vector direction measurement is available the resulting system is unobservable [2], [9] and the attitude can only be observed up to an unknown and unmeasurable transformation around the measured vector direction. More recent applications have considered attitude measurements in the inertial frame; for example, [16] uses multiple highly accurate GPS units and takes the difference between the outputs of two units as an inertial vector measurement of the body-fixed base-line between the GPS units on the airframe.

In this paper, we provide a detailed development of the design and robust stability analysis of nonlinear attitude observers based on a collection of vectorial measurements in either the body-fixed or inertial frames. We consider rigid-body kinematics in the body-fixed frame and term measurements in the body-fixed frame (of inertial reference directions) as *complementary* while we term measurements in the inertial frame (of body-fixed reference directions) as *compatible*. We consider firstly the case of multiple measurements, all of which lie in either the body-fixed (complementary measurements) or inertial frame (compatible measurements). Similar results have been considered in prior literature [9], [16], although in this paper we generalise these results to deal with the case where the reference direction is possibly time-varying. We then extend these results to design an observer that uses both inertial and body-fixed vectorial measurements of known time-varying references at the same time. This new observer provides a unifying design paradigm for attitude observers based on direct vectorial measurements. The proposed observer specialises to the earlier observers in the case where vector measurements are

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only available in a single frame of reference and inherits the earlier stability results in such a case. We prove stability of the new observer in the case where there are measurements in both the body-fixed and inertial frames. Finally, we consider the case of single vectorial measurements in more detail. This is the only case where the rigid-body attitude kinematics may be unobservable: each vectorial measurement provides two degrees of measurement and thus two or more non-collinear vector measurements provide four degrees of measurement over a state-space of dimension three ($\mathrm{SO}(3)$). In the case where a single measurement is made in the body-fixed-frame *of a constant reference direction in the inertial frame*, the system is indeed unobservable, as was shown implicitly in Mahony *et al.* [9]. In this paper we prove the somewhat surprising result that if the measurement is made in the inertial frame of a constant direction in the body-fixed frame, and if the system itself is persistently exciting, then the observer is asymptotically stable, and consequently the system is observable. Finally, we consider the single vectorial measurement case considered in [9], however, we add the assumption that the inertial reference direction is itself *time-varying with persistently exciting motion*. We show that the persistence of excitation is sufficient to overcome the loss of observability seen in [9, Cor. 5.2] and prove asymptotic convergence of the observer. Thus, the only case where the proposed unified observer is not asymptotically stable is where a single vectorial measurement is made in the body-fixed-frame of a *stationary* reference direction in the inertial frame.

The body of the paper consists of three sections followed by a conclusion. Section II provides some initial definitions and notation as well as recapping the previously considered results. Section III provides the formulation of the observer combining body-fixed and inertial measurements. Section IV provides a detailed analysis of the case where only a single measurement is available and Section VI provides a short conclusion.

II. Preliminary material

A. Notation and mathematical identities

The Special Orthogonal Group is denoted $\mathrm{SO}(3)$. The associated Lie-algebra is the set of anti-symmetric matrices

$$\mathfrak{so}(3) = \{A \in \mathbb{R}^{3\times 3} \mid A = -A^\top\}.$$

For any two matrices $A, B \in \mathbb{R}^{n\times n}$ the Lie-bracket is $[A, B] = AB - BA$. Let $\Omega = (\Omega_1, \Omega_2, \Omega_3)^\top \in \mathbb{R}^3$ and define

$$\Omega_\times = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

For any $v \in \mathbb{R}^3$ then $\Omega_\times v = \Omega \times v$ is the vector cross product. The operator $\mathrm{vex} : \mathfrak{so}(3) \to \mathbb{R}^3$ denotes the inverse of the $(.)_\times$ operator:

$$\mathrm{vex}\,(\Omega_\times) = \Omega, \qquad \Omega \in \mathbb{R}^3,$$
$$\mathrm{vex}(A)_\times = A, \qquad A \in \mathfrak{so}(3).$$

For any two matrices $A, B \in \mathbb{R}^{n\times n}$ the Euclidean matrix inner product and Frobenius norm are defined as

$$\langle\langle A, B\rangle\rangle = \mathrm{tr}(A^\top B) \quad \text{and} \quad ||A|| = \sqrt{\langle\langle A, A\rangle\rangle}.$$

The following identities are used throughout the paper:

$$(Rv)_\times = Rv_\times R^\top, \qquad v_\times w = -w_\times v,$$
$$v_\times w_\times = wv^\top - (v^\top w)\cdot I_3, \qquad (v_\times)^\top v_\times = I_3 - vv^\top,$$
$$(v \times w)_\times = [v_\times, w_\times] = wv^\top - vw^\top,$$
$$v^\top w = \langle v, w\rangle = \frac{1}{2}\langle\langle v_\times, w_\times\rangle\rangle,$$
$$v^\top v = |v|^2 = \frac{1}{2}||v_\times||^2,$$

where $v, w \in \mathbb{R}^3$ and $R \in \mathrm{SO}(3)$. The following notation for frames of reference is used

- $\{A\}$ denotes an inertial (fixed) frame of reference.
- $\{B\}$ denotes a body-fixed frame of reference.
- $\{E\}$ denotes the estimator frame of reference.

Let $\mathbb{P}_a, \mathbb{P}_s$ denote, respectively, the anti-symmetric and symmetric projection operators in square matrix space

$$\mathbb{P}_a(H) = \frac{1}{2}(H - H^\top), \qquad \mathbb{P}_s(H) = \frac{1}{2}(H + H^\top).$$

We use two different errors between a true orientation R and an estimate $\hat{R}$ on $\mathrm{SO}(3)$

$$\bar{R} = \hat{R}R^\top, \qquad \tilde{R} = \hat{R}^\top R. \tag{1}$$

B. Complementary observer

The system considered is the kinematics of the orientation matrix

$$\dot{R} = R\Omega_\times, \tag{2}$$

where the angular velocity Ω is assumed known, e.g. from onboard gyrometer systems. Let $S^2_{\{B\}}$ denote the unit sphere in the body-fixed-frame $\{B\}$. Consider multiple measurements $a_i \in S^2_{\{B\}}$ associated with several ($n \geq 1$) reference directions $a_{0i} \in S^2_{\{A\}}$, $i = 1, 2, \ldots n$ expressed as vectors on the unit sphere in the inertial frame $\{A\}$. Let $\hat{a}_i$ be the estimates

$$\hat{a}_i = \hat{R}^\top a_{0i}.$$

An example of such a measurement is the earth's magnetic field measured by magnetometers with known orientation with respect to the body-fixed frame $\{B\}$. The orientation of the magnetic field with respect to the inertial frame $\{A\}$ must be known *a priori*. A second common example is an estimate of the gravitational vector field measured by accelerometers, although this measurement may be corrupted by ego motion of the IMU.

In most applications the known reference directions $a_{0i} \in S^2_{\{A\}}$ are constant vectors. In this paper, however, we will also consider time-varying vector directions

$$a_{0i} := a_{0i}(t) \in S^2_{\{A\}}$$

to provide a more general development. We emphasise that the value of $a_{0i}(t) \in \{A\}$ must be known *a priori* at all times. The following definition will be useful in the sequel.

Definition 2.1: A set of $n \geq 2$ time-varying vector directions $v_i(t) \in S^2$, $i = 1, 2, \ldots, n$ is called *uniformly non-collinear* if there exists a constant $c > 0$ such that for all times t

$$\min_{i \neq j} |v_i(t) \times v_j(t)| > c.$$

This means that at all times at least two of the vector directions form an angle bounded away from zero by a *uniform* positive constant. Note that a set of *constant* vector directions is uniformly non-collinear if and only if it is non-collinear. Note further that $\{v_{i0}(t)\}$ uniformly non-collinear implies $\{v_i(t)\}$ uniformly non-collinear for any time-varying $R(t) \in \mathrm{SO}(3)$ and $v_i(t) = R(t)v_{i0}(t)$. Moreover, in this case the weighted sum of projectors $Q := \sum_{i=1}^n k_i \pi_{v_i} = \sum_{i=1}^n k_i(I - v_i v_i^\top)$ is positive definite for any choice of $k_i > 0$, $i = 1, 2, \ldots, n$ and, in fact, the minimum eigenvalue $\lambda_{\min}(Q)$ is positive and uniformly bounded away from zero for all times t.

The complementary observer proposed in [9] is

$$\dot{\hat{R}} = \hat{R}(\Omega + \alpha)_\times; \quad \text{where} \quad \alpha = \sum_{i=1}^n k_i(a_i \times \hat{a}_i). \tag{3}$$

Proposition 2.2: Consider the rotation kinematics Eq. 2. Assume that there are two or more ($n \geq 2$) vectorial measurements $a_i \in S^2_{\{B\}}$ available, associated with n uniformly non-collinear directions $a_{0i}(t) \in S^2_{\{A\}}$. Consider the complementary observer given by Eq. 3. Let $\bar{R} = \hat{R}R^\top$ denote the observer/plant error as an element on $\mathrm{SO}(3)$. Then the equilibrium $\bar{R} = I_3$ is locally exponentially stable with a basin of attraction including at least all initial conditions such that $\mathrm{tr}(I_3 - \bar{R}) < 4$.

Proof: The proof of this result follows directly from [9, Thm. 5.1]. In [9] the reference directions were assumed to be stationary, however, the proof did not use this assumption and applies directly to the case of uniformly non-collinear time-varying reference signals. ■

Note that $0 \leq \mathrm{tr}(I_3 - \bar{R}) \leq 4$ for $\bar{R} \in \mathrm{SO}(3)$ and that $\{\bar{R} \,|\, \mathrm{tr}(I_3 - \bar{R}) = 4\}$ has measure zero in $\mathrm{SO}(3)$. Hence Proposition 2.2 is in fact an almost global result.

C. Compatible observer

Unlike the complementary observer, for the compatible observer the measurements are associated with n body-fixed directions. Consider multiple measurements $b_i \in S^2_{\{A\}}$ associated with several ($n \geq 1$) possibly time-varying directions $b_{0i} := b_{0i}(t) \in S^2_{\{B\}}$, $i = 1, 2, \ldots, n$. Let $\hat{b}_i$ be the estimates

$$\hat{b}_i = \hat{R}b_{0i}.$$

Such measurements are less common than those discussed in the previous subsection. An example of such a measurement is the differential vector derived from comparing two inertial position measurements obtained from separate GPS units with a known base-line in the body-fixed frame $\{B\}$ [16]. Another manner in which such measurements can be obtained is by inertial fixed sensor systems that transmit measurements to an onboard attitude observer. Again, the angular velocity Ω is assumed known.

Proposition 2.3: Consider the rotation kinematics Eq. 2. Assume that there are two or more ($n \geq 2$) vectorial measurements $b_i \in S^2_{\{A\}}$ available, associated with n uniformly non-collinear directions $b_{0i}(t) \in S^2_{\{B\}}$. Consider the following compatible observer

$$\dot{\hat{R}} = \hat{R}(\Omega + \hat{R}^\top \beta)_\times; \quad \text{where} \quad \beta = \sum_{i=1}^n k_i(\hat{b}_i \times b_i). \tag{4}$$

Let $\bar{R} = \hat{R}R^\top$ denote the observer/plant error as an element on $\mathrm{SO}(3)$. Then the equilibrium $\bar{R} = I_3$ is locally exponentially stable with a basin of attraction including at least all initial conditions such that $\mathrm{tr}(I_3 - \bar{R}) < 4$.

Proof: (See also [16].) Recall the dynamics of the compatible observer given by Eq. 4,

$$\begin{aligned} \dot{\hat{R}} &= \hat{R}(\Omega + \hat{R}^\top \beta)_\times = \hat{R}R^\top \omega_\times R + \beta_\times \hat{R} \\ &= \hat{R}R^\top \omega_\times R\hat{R}^\top \hat{R} + \beta_\times \hat{R} = (\bar{R}\omega + \beta)_\times \hat{R}. \end{aligned} \tag{5}$$

Here $w = R\Omega$. Differentiating $\bar{R} = \hat{R}R^\top$ one obtains

$$\dot{\bar{R}} = (\bar{R}\omega + \beta)_\times \bar{R} - \bar{R}\omega_\times = \beta_\times \bar{R}.$$

Consider the following candidate Lyapunov function

$$E = \mathrm{tr}(I_3 - \bar{R}). \tag{6}$$

Differentiating E along the solutions of the system, one obtains

$$\dot{E} = -\mathrm{tr}(\beta_\times \bar{R}).$$

Recalling Eq. 4, the matrix identities listed in Section II-A, and noting that $\hat{b}_i = \hat{R}b_{0i} = \bar{R}b_i$ for $i = 1, 2, \ldots, n$, one obtains

$$\beta_\times = \sum_{i=1}^n k_i(\hat{b}_i \times b_i)_\times = \sum_{i=1}^n k_i(b_i b_i^\top \bar{R}^\top - \bar{R}b_i b_i^\top). \tag{7}$$

Therefore, the derivative of E may be written as

$$\begin{aligned} \dot{E} &= -\mathrm{tr}\left(\sum_{i=1}^n k_i(b_i b_i^\top \bar{R}^\top - \bar{R}b_i b_i^\top)\bar{R}\right) \\ &= -\sum_{i=1}^n k_i b_i^\top (I - \bar{R}\bar{R})b_i. \end{aligned} \tag{8}$$

We have

$$\begin{aligned} b_i^\top \mathbb{P}_a(\bar{R})\mathbb{P}_a(\bar{R})b_i &= \frac{1}{4}b_i^\top (\bar{R} - \bar{R}^\top)(\bar{R} - \bar{R}^\top)b_i \\ &= \frac{1}{4}\left(b_i^\top (\bar{R}\bar{R} - 2I_3)b_i + b_i^\top (\bar{R}^\top \bar{R}^\top)b_i\right) \\ &= \frac{1}{4}\left(b_i^\top (\bar{R}\bar{R} - 2I_3)b_i + b_i^\top (\bar{R}\bar{R})b_i\right) \\ &= -\frac{1}{2}b_i^\top (I - \bar{R}\bar{R})b_i \end{aligned}$$

and, again using the identities listed in Section II-A,

$$\mathbb{P}_a(\bar{R})b_i = -(b_i)_\times \mathrm{vex}\left(\mathbb{P}_a(\bar{R})\right).$$

Hence Eq. 8 implies

$$\begin{aligned} \dot{E} &= -2\mathrm{vex}\left(\mathbb{P}_a(\bar{R})\right)^\top Q \ \mathrm{vex}\left(\mathbb{P}_a(\bar{R})\right) \\ &\leq -\lambda_{\min}(Q)||\mathbb{P}_a(\bar{R})||^2, \end{aligned} \tag{9}$$

where $Q = \sum_{i=1}^n k_i (b_i)_\times^\top (b_i)_\times = \sum_{i=1}^n k_i (I_3 - b_i b_i^\top) = \sum_{i=1}^n k_i \pi_{b_i}$. Since we assume $\{b_{0i}(t)\}$ to be uniformly non-collinear and since $b_i = R b_{0i}$, the eigenvalue $\lambda_{\min}(Q)$ is uniformly bounded away from zero and the error converges such that $||\mathbb{P}_a(\bar{R})|| \to 0$. The condition $||\mathbb{P}_a(\bar{R})|| = 0$ implies either $\bar{R} = I_3$ or $\bar{R} = U^\top \mathrm{diag}(1,-1,-1) U$ for some $U \in \mathrm{SO}(3)$. The first case is an isolated minimum of the Lyapunov function and local exponential convergence follows. The second case corresponds to $\mathrm{tr}(I - \bar{R}) = 4$ and is a maximum of the Lyapunov function. Since $||\mathbb{P}_a(\bar{R})|| \neq 0$ implies $E(t) < E(0)$ for $t > 0$ this completes the proof. ■

III. Combined observer (compatible and complementary)

In this section we consider the situation where the available measurements are associated with n_1 body-fixed reference directions ($n_1 \geq 1$) and n_2 inertial reference directions ($n_2 \geq 1$). Let $\hat{b}_i$ be the estimates

$$\hat{b}_i = \hat{R} b_{0i},\ i = 1, 2, \ldots, n_1,$$

and let $\hat{a}_j$ be the estimates

$$\hat{a}_j = \hat{R}^\top a_{0j}, j = 1, 2, \ldots, n_2.$$

The following result provides a unified framework for the design of nonlinear observers for combinations of compatible and complementary measurements. Once again the angular velocity Ω is assumed known.

Theorem 3.1: Consider the rotation kinematics Eq. 2. Assume that there are one or more ($n_1 \geq 1$) compatible measurements b_i and one or more ($n_2 \geq 1$) complementary measurements a_j available. Assume that at least one pair of directions out of $\{R b_{0i}\} \cup \{a_{0j}\}$ is uniformly non-collinear. Consider the following observer

$$\dot{\hat{R}} = \hat{R}(\Omega + \hat{R}^\top \beta + \alpha)_\times, \tag{10}$$

where the innovation terms α and β are given by Eqs. 3 and 4 respectively. Then the equilibrium $\bar{R} = I_3$ is locally exponentially stable with a basin of attraction including at least all initial conditions such that $\mathrm{tr}(I_3 - \bar{R}) < 4$.

Proof: Differentiating $\bar{R} = \hat{R} R^\top$ yields

$$\dot{\bar{R}} = \beta_\times \bar{R} + (\hat{R}\alpha)_\times \bar{R}.$$

Recall the candidate Lyapunov function $E = \mathrm{tr}(I_3 - \bar{R})$. Differentiating yields

$$\begin{aligned}
\dot{E} &= -\mathrm{tr}(\beta_\times \bar{R}) - \mathrm{tr}((\hat{R}\alpha)_\times \bar{R}) \\
&= -\mathrm{tr}(\beta_\times \bar{R}) - \mathrm{tr}(\hat{R}\alpha_\times R^\top) \\
&= -\mathrm{tr}(\beta_\times \bar{R}) - \mathrm{tr}(\alpha_\times \tilde{R}^\top) \\
&= -\mathrm{tr}(\beta_\times \bar{R}) + \mathrm{tr}(\alpha_\times \tilde{R}),
\end{aligned}$$

where $\tilde{R} = \hat{R}^\top R$. Analogously to the computation provided in the previous section, the derivative of E can be expressed as follows:

$$\dot{E} = -2 \mathrm{vex} \mathbb{P}_a(\bar{R})^\top Q_0 \mathrm{vex} \mathbb{P}_a(\bar{R}),$$

where $Q_0 = \sum_{i=1}^{n_1} k_i \pi_{b_i} + \sum_{j=1}^{n_2} k_j \pi_{a_{0j}}$. Using the fact that there are at least two uniformly non-collinear directions, it follows that

$$\dot{E} \leq -\lambda_{\min}(Q_0) ||\mathbb{P}_a(\bar{R})||^2.$$

The statement now follows as in the proof of Proposition 2.3. ■

IV. Attitude observer using a single measurement

In this section we consider the situation when only a single direction is measured ($n = n_1 + n_2 = 1$). This situation is significantly more complicated than the previous section. In particular, the stability proofs in the previous section relied on the positive definiteness of

$$Q_0 = \sum_{i=1}^{n_1} k_i \pi_{b_i} + \sum_{j=1}^{n_2} k_j \pi_{a_{0j}},$$

where each $\pi_\star$ is a rank 2 semi-definite projection matrix. Generically, for two or more directions, either inertial or body-fixed, the resulting Q_0 is positive definite and exponential stability results. In the case of a single measurement, this is clearly no longer the case. In fact, a single vector measurement is the only case where it is possible that the system is unobservable, since two or more non-collinear measurements provide four degrees of measurement over a state-space of dimension three ($\mathrm{SO}(3)$). In the case where a single measurement is made in the body-fixed frame *of a constant reference direction in the inertial frame*, the system is indeed unobservable [9]. However, in other cases the system is observable, at least as long as there is a persistence of excitation in the system variables. The earlier proofs of stability clearly do not apply in this case and it is necessary to use a more sophisticated argument.

Let b (resp. a) be a measured direction in the inertial frame (resp. in the body-fixed frame) associated with a known reference direction $b_0(t) \in \{B\}$ (resp. $a_0(t) \in \{A\}$) that may be time-varying. One has $b = R b_0$ (resp. $a = R^\top a_0$). Consider the estimate $\hat{b} = \hat{R} b_0$ (resp. $\hat{a} = \hat{R}^\top a_0$). Assume that

$$\dot{b}_0 = \Lambda^b_\times b_0 \text{ and } \dot{a}_0 = \Lambda^a_\times a_0, \tag{11}$$

where Λ^b (resp. Λ^a) represents the orientation velocity of b_0 (resp. a_0) with respect to the body-fixed frame (resp. inertial frame). As always in this paper the angular velocity Ω is assumed known.

Theorem 4.1: Consider the rotation kinematics Eq. 2 along with the observer given by Eq. 3. Let b be a measured direction in the inertial frame associated with a known direction in the body-fixed frame b_0 such that $\dot{b}_0 = \Lambda^b_\times b_0$. Assume that the orientation velocities Ω and Λ^b are bounded continuous signals. If $\bar{\Omega} = \Omega + \Lambda^b$ is persistently exciting[1] then the error $\bar{R}$ is locally asymptotically stable around the equilibrium I_3, with a basin of attraction containing at least all initial conditions such that $\mathrm{tr}(I_3 - \bar{R}) < 4$.

[1]We call $\bar{\Omega}$ *persistently exciting* if there exist T, $\mu > 0$ such that $|\int_t^{t+T} \bar{\Omega}_\times b_0 \, \mathrm{d}t| \geq \mu$ for all t.

Proof: Recall the dynamics of the compatible observer in the case where $n = n_1 = 1$,

$$\dot{\hat{R}} = \hat{R}(\Omega + \hat{R}^\top \beta)_\times; \quad \text{where} \quad \beta = k(\hat{b} \times b).$$

Consider the following candidate Lyapunov function

$$E = \text{tr}(I_3 - \bar{R}).$$

As in the proof of Proposition 2.3 it follows

$$\dot{E} = -k(1 - b^\top \bar{R}^2 b) = -k(1 - b_0^\top \tilde{R}^2 b_0) \leq 0.$$

Since $\dot{E}$ is negative semi-definite, the candidate Lyapunov function is bounded with $0 \leq E(t) \leq E(0)$ for $t \geq 0$. Additionally, $\ddot{E}$ is bounded[2] and hence $\dot{E}$ is uniformly continuous. An application of Barbalat's lemma yields $b^\top \bar{R}^2 b \to 1$. This implies that $\bar{R}b = b$ since the alternative possibility, $\bar{R}b = -b$, is excluded by the constraint $\text{tr}(I_3 - \bar{R}) < 4$ on the initial condition and the fact that $E(t) \leq E(0)$. Consequently, $\hat{b}$ converges asymptotically to b.

It remains to show that $\bar{R}$ converges to I_3. Starting with the fact that $\hat{b} \to b$, one sees that

$$\tilde{R}b_0 - b_0 \to 0.$$

Differentiating this equation and recalling the fact that Ω and Λ are uniformly continuous and bounded signals, we get (again using Barbalat's lemma)

$$-(\Omega - \tilde{R}\Omega)_\times b_0 + \tilde{R}\Lambda^b_\times b_0 - \Lambda^b_\times b_0 \to 0.$$

Given that $\tilde{R}\Lambda^b_\times b_0 = (\tilde{R}\Lambda^b)_\times \tilde{R}b_0 \to (\tilde{R}\Lambda^b)_\times b_0$, the above expression may be rewritten as follows:

$$-(\bar{\Omega} - \tilde{R}\bar{\Omega})_\times b_0 \to 0.$$

Consequently, in the limit $\tilde{R}$ is a solution to the equation

$$\tilde{R}\bar{\Omega} = \bar{\Omega}.$$

This implies that the vector $\bar{\Omega}$ is an eigenvector associated with the eigenvalue $\lambda = 1$ of the rotation matrix $\tilde{R}$. Given that b_0 is another eigenvector associated with the eigenvalue $\lambda = 1$, and that the eigenvalues of an orthogonal matrix are of the form[3]

$$\text{eig}(\tilde{R}) = (1,\ \cos(\theta) + i\sin(\theta),\ \cos(\theta) - i\sin(\theta)),$$

it follows that, as long as $\bar{\Omega}$ is persistently exciting, $\tilde{R}$ converges to I_3. ■

The second case concerns the situation where the measurement is made in the body-fixed frame. In this case, if the associated inertial reference direction is constant then the system is unobservable and the best convergence for the error dynamics that can be obtained is to the set

$$\mathcal{S}_{b_0} = \{\tilde{R} \,|\, \tilde{R}b_0 = b_0\}$$

[9, Cor. 5.2]. The following result shows that this is a special case. In particular, if the inertial reference direction is itself time-varying and persistently exciting then one recovers almost-global stability of the observer error system.

Theorem 4.2: Consider the rotation kinematics Eq. 2 along with the observer given by Eq. 4. Let a be a measured direction in the body-fixed frame associated with a known direction in the inertial frame a_0 such that $\dot{a}_0 = \Lambda^a_\times a_0$. Assume that the orientation velocities Ω and Λ^a are bounded continuous signals. If Λ^a is persistently exciting[4] then the error $\bar{R}$ is locally asymptotically stable around the equilibrium I_3, with a basin of attraction including at least all initial conditions such that $\text{tr}(I_3 - \bar{R}) < 4$.

Proof: The proof of this theorem is analogous to the previous one. First, recall the dynamics of the complementary observer in the case where $n = n_2 = 1$,

$$\dot{\hat{R}} = \hat{R}(\Omega + \alpha)_\times; \quad \text{where} \quad \alpha = k(a \times \hat{a}).$$

Consider the following candidate Lyapunov function

$$E = \text{tr}(I_3 - \bar{R}).$$

It is straightforward to verify that:

$$\dot{E} = -k(1 - a^\top \tilde{R}^2 a) = -k(1 - a_0^\top \bar{R}^2 a_0) \leq 0.$$

Given that E and $\ddot{E}$ are bounded, Barbalat's lemma ensures that $a^\top \tilde{R}^2 a$ (or, equivalently, $a_0^\top \bar{R} a_0$) converges asymptotically to 1.

This implies that $\tilde{R}a = a$ since the alternative possibility, $\tilde{R}a = -a$, is excluded by the constraint $\text{tr}(I_3 - \bar{R}) < 4$ on the initial condition and the fact that $E(t) \leq E(0)$. Consequently, $\hat{a}$ converges asymptotically to a.

It remains to show that $\tilde{R}$ converges to I_3. Since $a \to \hat{a}$, one obtains

$$\bar{R}a_0 - a_0 \to 0.$$

Differentiating this equation and recalling the fact that Λ^a is a uniformly continuous and bounded signal, we obtain (again using Barbalat's lemma)

$$\bar{R}\Lambda^a_\times a_0 - \Lambda^a_\times a_0 \to 0.$$

Given that $\bar{R}\Lambda^a_\times a_0 = (\bar{R}\Lambda^a)_\times \bar{R}a_0 \to (\tilde{R}\Lambda^a)_\times a_0$, the above expression may be rewritten as follows

$$-(\Lambda^a - \bar{R}\Lambda^a)_\times a_0 \to 0.$$

Consequently, in the limit $\bar{R}$ is a solution to the equation

$$\bar{R}\Lambda^a = \Lambda^a.$$

This implies that the vector Λ^a is an eigenvector associated with the eigenvalue $\lambda = 1$. As long as the persistence of excitation condition on is satisfied by Λ^a, it follows that $\bar{R}$ converges to I_3. ■

[2] Using the fact that the orientation velocities Ω and Λ^b are bounded continuous signals, it is straightforward to show that $\ddot{E}$ is bounded.

[3] θ is the angle from the angle-axis representation.

[4] We call Λ_a *persistently exciting* if there exist T, $\mu > 0$ such that $|\int_t^{t+T} \Lambda^a_\times a_0 \,\mathrm{d}t| \geq \mu$ for all t.

V. Simulation results

In this section some simulation results are presented. Only the situation where the measurement of a single direction (constant or time-varying) is considered in either the body-fixed frame or in the inertial frame. For all cases considered the initial deviation $\bar{R}$ is chosen large, $\Omega = (1\ -2\ 1)^\top$ and $k = 10$. For the situation where the measured direction is constant a_0 (resp. b_0) $= (0\ 0\ 1)^\top$. When the measured direction is time varying, a_0 (resp. b_0) $= (\frac{1}{2}\sin t\ \frac{1}{2}\cos t\ \frac{1}{\sqrt{2}})^\top$. Figure 1 shows clearly that, except for the complementary

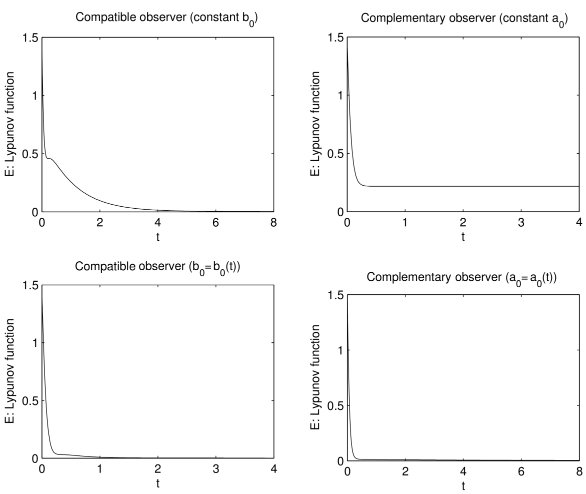

Fig. 1. Compatible and complementary observers using a single measurement.

observer along with the measurement of a constant direction a_0 for which the system is unobservable, the system is observable and the asymptotic convergence of $\bar{R}$ towards I_3 is guaranteed.

VI. Conclusion

In this paper, we provided a detailed development of the design and robust stability analysis of nonlinear attitude observers based on a collection of vectorial measurements in either the body-fixed or inertial frames. The main results in the paper are the analysis of the single vectorial measurement cases and the somewhat surprising result that an inertial measurement leads to asymptotically stable observers (at least assuming persistence of excitation of the of rigid-body motion). The secondary result that body-fixed measurements of time-varying inertial reference vectors also lead to stable observer error dynamics is also of considerable interest. Unfortunately, a common case encountered in practice involves body-fixed measurements of inertial constant reference vectors (eg. body-fixed measurements of gravitational or magnetic field directions), the case where only partial stability can be shown. There are, however, an increasing number of emerging applications where the results in this paper may be highly applicable. In particular, we mention applications involving GPS sensors for mobile vehicle applications used to obtain attitude reference measurements.

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References

[1] N. Aghannan and P. Rouchon. An intrinsic observer for a class of Lagrangian systems. *IEEE Trans. Automat. Contr.*, 48(6):936–945, June 2003.

[2] S. Bonnabel, Ph. Martin, and P. Rouchon. Symmetry-preserving observers. *IEEE Trans. Automat. Contr.*, 53(11):2514–2526, 2008.

[3] S. Bonnabel and P. Rouchon. *Control and Observer Design for Nonlinear Finite and Infinite Dimensional Systems*, volume 322 of *Lecture Notes in Control and Information Sciences*, chapter On Invariant Observers, pages 53–67. Springer-Verlag, 2005.

[4] D. Campoloa, F. Kellerb, and E. Guglielmellia. Inertial/magnetic sensors based orientation tracking on the group of rigid body rotations with application to wearable devices. In *Proc. IEEE/RSJ Int. Conf. IROS*, pages 4762–4767, Beijing, China, October 9 - 15 2006.

[5] J.L. Crassidis, F.L. Markley, and Y. Cheng. Nonlinear attitude filtering methods. *Journal of Guidance, Control, and Dynamics*, 30(1):12–28, January 2007.

[6] O. Egland and J.M. Godhaven. Passivity-based adaptive attitude control of a rigid spacecraft. *IEEE Trans. Automat. Contr.*, 39:842–846, April 1994.

[7] T. Hamel and R. Mahony. Attitude estimation on $mathrmSO(3)$ based on direct inertial measurements. In *Proc. IEEE ICRA*, pages 2170–2175, Orlando Fl., USA, April 2006.

[8] R. Hatch. Synergism of GPS code and carrier measurements. In *Proceedings of the 3rd International Geodetic Sypomsium on Satellite Doppler Positioning*, volume 2, pages 1213–1232, Las Cruces, New Mexico, February 1982.

[9] R. Mahony, T. Hamel, and J.-M. Pflimlin. Non-linear complementary filters on the special orthogonal group. *IEEE Trans. Automat. Contr.*, 53(5):1203–1218, June 2008.

[10] D.H.S. Maithripala, W.P. Dayawansa, and J.M. Berg. Intrinsic observer-based stabilization for simple mechanical systems on Lie-groups. *SIAM J. Control and Optim.*, 44(5):1691–1711, 2005.

[11] Ph. Martin and E. Salaün. Invariant observers for attitude and heading estimation from low-cost inertial and magnetic sensors. In *Proc. IEEE CDC*, pages 1039–1045, 2007.

[12] Ph. Martin and E. Salaün. An invariant observer for earth-velocity-aided attitude heading reference systems. In *Proc. IFAC World Congress*, Seoul, Korea, July 6-11 2008.

[13] S. Salcudean. A globally convergent angular velocity observer for rigid body motion. *IEEE Trans. Automat. Contr.*, 46, no 12:1493–1497, 1991.

[14] A. Tayebi and S. McGilvray. Attitude stabilization of a VTOL quadrotor aircraft. *IEEE Trans. Contr. Sys. Techn.*, 14(3):562–571, May 2006.

[15] J. Thienel and R. M. Sanner. A coupled nonlinear spacecraft attitude controller and observer with an unknow constant gyro bias and gyro noise. *IEEE Trans. Automat. Contr.*, 48(11):2011 – 2015, Nov. 2003.

[16] J.F. Vasconcelos, C. Silvestre, and P. Oliveira. A nonlinear GPS/IMU based observer for rigid body attitude and position estimation. In *Proc. IEEE CDC*, pages 1255–1260, 9-11 Dec. 2008.

[17] J.F. Vasconcelos, C. Silvestre, and P. Oliveira. A nonlinear observer for rigid body attitude estimation using vector observations. In *Proc. IFAC World Congress*, pages 8599–8604, Seoul, Korea, July 2008.

[18] B. Vik and T. Fossen. A nonlinear observer for GPS and INS integration. In *Proc. IEEE CDC*, pages 2956–2961, 2001.