

On Projective Reconstruction In Arbitrary Dimensions

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Abstract

We study the theory of projective reconstruction for multiple projections from an arbitrary dimensional projective space into lower-dimensional spaces. This problem is important due to its applications in the analysis of dynamical scenes. The current theory, due to Hartley and Schaffalitzky, is based on the Grassmann tensor, generalizing the ideas of fundamental matrix, trifocal tensor and quadrifocal tensor in the well-studied case of 3D to 2D projections. We present a theory whose point of departure is the projective equations rather than the Grassmann tensor. This is a better fit for the analysis of approaches such as bundle adjustment and projective factorization which seek to directly solve the projective equations. In a first step, we prove that there is a unique Grassmann tensor corresponding to each set of image points, a question that remained open in the work of Hartley and Schaffalitzky. Then, we prove that projective equivalence follows from the set of projective equations, provided that the depths are all nonzero. Finally, we demonstrate possible wrong solutions to the projective factorization problem, where not all the projective depths are restricted to be nonzero.

1. Introduction

In this paper we develop the theory of projective reconstruction for multiple projections from an arbitrary dimensional projective space to lower-dimensional spaces. A set of such projections can be represented as

$$\lambda_{ij}\mathbf{x}_{ij} = \mathbf{P}_i\mathbf{X}_j \quad (1)$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$, where $\mathbf{X}_j \in \mathbb{R}^r$ are high-dimensional (HD) points, $\mathbf{P}_i \in \mathbb{R}^{s_i \times r}$ are projection matrices, $\mathbf{x}_{ij} \in \mathbb{R}^{s_i}$ are image points and λ_{ij} -s are nonzero scalars known as *projective depths*. The problem of *projective reconstruction* is to obtain the projection matrices \mathbf{P}_i , the points \mathbf{X}_j and the depths λ_{ij} , up

to a projective ambiguity, given the image points \mathbf{x}_{ij} . The goal of this paper is to deduce the uniqueness of such reconstruction, given the set of equations (1).

The classic case of projections from 3D scenes to 2D images has been intensely studied in the past two decades [4]. In this case $\mathbf{X}_j \in \mathbb{R}^4$ resp. $\mathbf{x}_{ij} \in \mathbb{R}^3$ represent points in the projective spaces \mathbb{P}^3 resp. \mathbb{P}^2 in homogeneous coordinates. When the scene is rigid, the traditional way of analysing and solving the problem of projective reconstruction is via the bifocal tensor (fundamental matrix), trifocal tensor or quadrifocal tensor [4, 5]. The standard procedure is to build a multiple view tensor from point (or line) correspondences among two, three or four views, and then from the tensor extract the original camera matrices up to projectivity. The 3D points can be subsequently estimated through a triangulation procedure.

The recovery of structure and motion is more challenging in the case of nonrigid motions. Wolf and Shashua in [11] consider a number of different structure and motion problems in which the scene observed by a perspective camera is nonrigid. They show that all the given problems can be modeled as projections from a higher-dimensional projective space \mathbb{P}^k into \mathbb{P}^2 for $k = 3, 4, 5, 6$. They use tensorial approaches to address each of the problems. Hartley and Vidal [2] considered the problem of perspective nonrigid deformation, assuming that the scene deforms as a linear combination of k different linearly independent basis shapes. They show that the problem can be modeled as projections from \mathbb{P}^{3k} to \mathbb{P}^2 .

Such applications manifest the need for a general theory of projective reconstruction for arbitrary dimensional spaces. Hartley and Schaffalitzky [3] present a comprehensive theory to address the projective reconstruction for general projections. Their theory unifies the previous work by introducing the Grassmann tensor, which generalizes the concepts of bifocal, trifocal and quadrifocal tensors used in $\mathbb{P}^3 \rightarrow \mathbb{P}^2$ projections, and other tensors used for special cases in other dimensions. The central theorem in [3] proves that the

projection matrices can be obtained up to projectivity from the corresponding Grassmann tensor.

Tensor-based projective reconstruction is sometimes not accurate enough, especially in the presence of noise. One problem is imposing necessary restrictions on the form of the Grassmann tensor during its computation. As a simple example the fundamental matrix (bifocal tensor) needs to be of rank 2. The other restriction is that for projections from \mathbb{P}^{r-1} , at most r views contribute to the computation of each tensor. This prevents us from making use of the whole set of image points to reduce the estimation error. This has led to the use of other approaches such as bundle adjustment [10] and projective factorization [9, 6, 8, 1], in which the projection equations (1) are directly solved for projection matrices P_i , HD points \mathbf{X}_j and projective depths λ_{ij} . Analysing such methods requires a theory which derives the projective reconstruction from the projection equations (1), rather than from the Grassmann tensor.

Recently, Nasihatkon et al. developed such a theory for $\mathbb{P}^3 \rightarrow \mathbb{P}^2$ projections [7]. They classify all the wrong solutions to the projective factorization equation $\Lambda \odot [\mathbf{x}_{ij}] = \mathbf{P}\mathbf{X}$ which is the matrix form of (1), based on the zero patterns of the depth matrix Λ . The theory is particularly useful in the design and verification of factorization-based algorithms. Their theory, however, is restricted to the case of 3D to 2D projections. As far as we are aware, no such theory has been proposed for general dimensional projections.

In this paper, we give a theory which deduces projective reconstruction from the set of equations (1). As a first step, we need to answer a question which is left open in [3], namely whether the set of image points \mathbf{x}_{ij} uniquely determine the Grassmann tensor, up to a scaling factor. Notice that this is important even for tensor-based projective reconstruction. Our theory in section 3.1 gives a positive answer to this question.

The second question is whether all configurations of projective matrices and HD points projecting into the same image points \mathbf{x}_{ij} (all satisfying (1) with nonzero depths λ_{ij}) are projectively equivalent. This is important for the analysis of bundle adjustment as well as factorization-based approaches. The answer to such a simple question is by no means trivial. Notice that the uniqueness of the Grassmann tensor is not sufficient for proving this, as it does not rule out the existence of degenerate solutions $\{P_i\}$ whose corresponding Grassmann tensor is zero. This paper gives a positive answer to this question as well, as a consequence of the theory presented in section 4.

The last issue, which only concerns the factorization-based approaches, is classifying all

the degenerate solutions to the projective factorization equation $\Lambda \odot [\mathbf{x}_{ij}] = \mathbf{P}\mathbf{X}$. This is the matrix form of (1) and shows the idea behind factorization methods: find the depths $\Lambda = [\lambda_{ij}]$ such that the matrix of weighted image points $\Lambda \odot [\mathbf{x}_{ij}] = [\lambda_{ij}\mathbf{x}_{ij}]$ has a rank- r factorization in the form of the product of two matrices \mathbf{P} and \mathbf{X} . In such algorithms it is difficult or inefficient to enforce all the depths λ_{ij} to be nonzero. Therefore, it is important to classify the projectively nonequivalent solutions which occur when not all elements of Λ are restricted to be nonzero. We study the form of such degenerate solutions in section 5, and demonstrate them by giving examples.

2. Background

2.1. Conventions

We borrow most of our notation from previous work, in particular from [4, 3, 7]. We use typewriter letters (\mathbf{A}) for matrices, bold letters (\mathbf{a}, \mathbf{A}) for vectors, lower-case normal letters (a) for scalars and upper-case normal letters (A) for sets, except for special sets like \mathbb{R} and \mathbb{P} . We use calligraphic letters (\mathcal{A}) for both tensors and mappings (functions). To refer to the column space and null space of a matrix \mathbf{A} we respectively use $\mathcal{C}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$. The vertical concatenation of a set of matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_m$ is denoted by $\text{stack}(\mathbf{A}_1, \dots, \mathbf{A}_m)$. We make use of the terms “generic” and “in general position” for entities such as points, matrices and subspaces. In such cases, if the generic properties are not explicitly stated, we simply mean that they belong to an open and dense subset which will be implicitly determined from the properties assumed as a consequence of genericity in our proofs.

Throughout the paper we deal with a set of high-dimensional (HD) points $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n \in \mathbb{R}^r$, which represent points in the projective space \mathbb{P}^{r-1} in homogeneous coordinates, and a set of projection matrices $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ where the i -th matrix $\mathbf{P}_i \in \mathbb{R}^{s_i \times r}$ represents a projective mapping from \mathbb{P}^{r-1} to \mathbb{P}^{s_i-1} . Each mapping takes each HD point \mathbf{X}_j to an image point $\mathbf{x}_{ij} \in \mathbb{R}^{s_i-1}$, through the relation $\lambda_{ij}\mathbf{x}_{ij} = \mathbf{P}_i\mathbf{X}_j$.

Here, the setup $(\{P_i\}, \{X_j\})$ is referred to as *the true configuration*. We also use a second setup of projection matrices and points $(\{\hat{P}_i\}, \{\hat{X}_j\})$. This new setup, denoted by *hatted* quantities, are referred to as the *estimated* configuration. The object of our main theorems here is to show that if the setup $(\{\hat{P}_i\}, \{\hat{X}_j\})$ projects into the same set of image points as $(\{P_i\}, \{X_j\})$, then $(\{\hat{P}_i\}, \{\hat{X}_j\})$ and $(\{P_i\}, \{X_j\})$ are projectively equivalent. We must stress that, here, the projection matrices P_i, \hat{P}_i , HD points X_j, \hat{X}_j and image points x_{ij} are treated as members of a real vector space, even though

they might represent quantities in a projective space. The equality sign “=” here is strict and never implies equality up to scale.

2.2. Projective equivalence

We formalize the concept of projective equivalence for HD points and projection matrices as follows.

Definition 1. Two sets of projection matrices $\{P_i\}$ and $\{\hat{P}_i\}$, with $P_i, \hat{P}_i \in \mathbb{R}^{s_i \times r}$ for $i = 1, 2, \dots, m$ are projectively equivalent if there exist nonzero scalars $\tau_1, \tau_2, \dots, \tau_m$ and an $r \times r$ invertible matrix H such that

$$\hat{P}_i = \tau_i P_i H, \quad i = 1, 2, \dots, m. \quad (2)$$

Two sets of points $\{X_j\}$ and $\{\hat{X}_j\}$ with $X_j, \hat{X}_j \in \mathbb{R}^r$ for $j = 1, 2, \dots, n$, are projectively equivalent if there exist nonzero scalars $\nu_1, \nu_2, \dots, \nu_n$ and an invertible $r \times r$ matrix G such that

$$\hat{X}_j = \nu_j G X_j, \quad j = 1, 2, \dots, n. \quad (3)$$

Two setups $(\{P_i\}, \{X_j\})$ and $(\{\hat{P}_i\}, \{\hat{X}_j\})$ are projectively equivalent if both (2) and (3) hold, and furthermore $G = H^{-1}$.

The following lemma is needed later on in the paper.

Lemma 1. Consider a set of points $X_1, X_2, \dots, X_n \in \mathbb{R}^r$ with $n > r$ with the generic properties

(P1) $\text{span}(X_1, \dots, X_n) = \mathbb{R}^r$, and

(P2) the set of points $\{X_i\}$ cannot be partitioned into $p \geq 2$ nonempty subsets, such that subspaces defined as the span of each subset are independent¹.

Now, for any set of points $\{\hat{X}_i\}$ projectively equivalent to $\{X_i\}$, the matrix G and scalars ν_j defined in (3) are unique up to a scale ambiguity of the form $(\beta G, \{\nu_j/\beta\})$ for any nonzero scalar β .

Notice that (P2) is generic only when $n > r$. The proof is based on the theory of eigenspaces of a square matrix and is given in the Supplementary Material.

2.3. Triangulation

The problem of Triangulation is to find a point X given its images through a set of known projections P_1, \dots, P_m . The next lemma provides conditions for uniqueness of triangulation.

Lemma 2 (Triangulation). Consider a set of projection matrices P_1, P_2, \dots, P_m with $P_i \in \mathbb{R}^{s_i \times r}$, and a point $X \in \mathbb{R}^r$, configured such that

¹Subspaces U_1, \dots, U_p are independent if $\dim(\sum_{j=1}^p U_j) = \sum_{j=1}^p \dim(U_j)$, where $\sum_{j=1}^p U_j = \{\sum_{j=1}^p u_j \mid u_j \in U_j\}$.

(T1) there does not exist any linear subspace of dimension less than or equal to 2, passing through X and nontrivially intersecting² all the null spaces $\mathcal{N}(P_1), \mathcal{N}(P_2), \dots, \mathcal{N}(P_m)$.

Now, for any nonzero $Y \neq 0$ in \mathbb{R}^r if the relations

$$P_i Y = \beta_i P_i X, \quad i = 1, 2, \dots, m \quad (4)$$

hold for scalars β_i , then $Y = \beta X$ for some scalar $\beta \neq 0$.

Notice that we have not assumed $\beta_i \neq 0$.

Proof. From $P_i Y = \beta_i P_i X$ we deduce

$$Y = \beta_i X + C_i \quad (5)$$

for some $C_i \in \mathcal{N}(P_i)$, which means $C_i \in \text{span}(X, Y)$. Now, if all C_i -s are nonzero, then the subspace $\text{span}(X, Y)$ nontrivially intersects all the subspaces $\mathcal{N}(P_i)$, $i = 1, \dots, m$, violating (T1). Hence, for some index k we must have $C_k = 0$. By (5), therefore, we have $Y = \beta_k X$, that is Y is equal to X up to scale. As Y is nonzero, β_k cannot be zero. \square

Notice that for the classic case of projections $\mathbb{P}^3 \rightarrow \mathbb{P}^2$, (T1) simply means that the camera centres $\mathcal{N}(P_i)$ and the projective point $\text{span}(X) \in \mathbb{P}^3$ are collinear. For general dimensional projections, however, it is not trivial to show that (T1) is generically true. A proof of this is provided in the Supplementary Material.

2.4. Valid profiles and the Grassmann tensor

Consider a set of projection matrices P_1, P_2, \dots, P_m , with $P_i \in \mathbb{R}^{s_i \times r}$, such that $\sum_{i=1}^m (s_i - 1) \geq r$. A valid profile [3] is defined as an m -tuple of nonnegative³ integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ such that $0 \leq \alpha_i \leq s_i - 1$ and $\sum \alpha_i = r$. Clearly, there might exist different valid profiles for a setup $\{P_i\}$. One can choose $r \times r$ submatrices of $P = \text{stack}(P_1, P_2, \dots, P_m)$ according to a profile α , by choosing α_i rows from each P_i . Notice that due to the property $\alpha_i \leq s_i - 1$, never the whole rows of any P_i is chosen for building the submatrix.

Consider the set of m index sets I_1, I_2, \dots, I_m , such that each I_i contains the indices of α_i rows of P_i . Each way of choosing I_1, I_2, \dots, I_m gives a square submatrix of $P = \text{stack}(P_1, \dots, P_m)$ where the rows of each P_i are chosen in order. The determinant of this submatrix is multiplied by a corresponding sign⁴ to form an entry of

²Two linear subspaces nontrivially intersect if their intersection has dimension one or more.

³Notice that, the definition of a valid profile here slightly differs from [3] which needs $\alpha_i \geq 1$. We choose this new definition for convenience, as it does not impose the restriction $m \leq r$ on the number of views.

⁴The sign is defined by $\prod_{i=1}^m \text{sign}(I_i)$ where $\text{sign}(I_i)$ is +1 or -1 depending on whether the sequence $(\text{sort}(I_i) \text{sort}(\bar{I}_i))$ is an even or odd permutation for $\bar{I}_i = \{1, \dots, s_i\} \setminus I_i$ (see [3]).

the Grassmann coordinate of $\mathbf{P} = \text{stack}(\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_m)$, shown here by $\mathcal{T}_{\alpha}^{I_1, I_2, \dots, I_m}$. Such entries for different choices of the I_i -s can be arranged in a multidimensional array \mathcal{T}_{α} called the Grassmann tensor corresponding to α . The dimension of \mathcal{T}_{α} is equal to the number of nonzero entries of $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, as \mathcal{T}_{α} does not depend on those matrices \mathbf{P}_i with $\alpha_i = 0$. To show the dependence of the Grassmann tensor on projection matrices \mathbf{P}_i , we sometimes use the mapping \mathcal{G}_{α} which takes a set of projection matrices to the corresponding Grassmann tensor, that is $\mathcal{T}_{\alpha} = \mathcal{G}_{\alpha}(\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_m)$. Notice that \mathcal{G}_{α} itself is not a tensor. Obviously, $\mathcal{G}_{\alpha}(\mathbf{P}_1, \dots, \mathbf{P}_m)$ is nonzero if and only if \mathbf{P} has a non-singular submatrix chosen according to α .

Hartley and Schaffalitzky [3] show that if a point \mathbf{X} is projected through $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_m$ into the image points $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, according to $\lambda_i \mathbf{x}_i = \mathbf{P}_i \mathbf{X}$, then for any set of full-column-rank matrices $\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_m$ such that $\mathbf{U}_i \in \mathbb{R}^{s_i \times (s_i - \alpha_i)}$ and $\mathbf{x}_i \in U_i = \mathcal{C}(\mathbf{U}_i)$, the following holds

$$\sum_{I_1, \dots, I_m} \mathcal{T}_{\alpha}^{I_1, I_2, \dots, I_m} \prod_{i=1}^m \det(\mathbf{U}_i^{\bar{I}_i}) = 0, \quad (6)$$

where $\mathbf{U}_i^{\bar{I}_i}$ is the square submatrix made by choosing rows of \mathbf{U}_i according to \bar{I}_i , the complement of I_i . Notice that $\det(\mathbf{U}_i^{\bar{I}_i})$ for different values of \bar{I}_i form the Grassmann coordinates of the subspace $U_i = \mathcal{C}(\mathbf{U}_i)$. The main theorem of [3] states that the projection matrices \mathbf{P}_i can be uniquely constructed from the Grassmann tensor, up to projectivity:

Theorem 1 ([3]). *Consider a set of m generic projection matrices $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_m$, with $\mathbf{P}_i \in \mathbb{R}^{s_i \times r}$, such that $m \leq r \leq \sum_i s_i - m$, and a valid profile $(\alpha_1, \alpha_2, \dots, \alpha_m)$ for which $\alpha_i \geq 1$ for all i . Then if at least one s_i is greater than 2, the matrices \mathbf{P}_i are determined up to a projective ambiguity from the set of minors of \mathbf{P} chosen with α_i rows from each \mathbf{P}_i . If $s_i = 2$ for all i , there are two equivalence classes of solutions.*

The constructive proof given by [3] gives a procedure to construct the projection matrices \mathbf{P}_i from the Grassmann tensor. From each set of image point correspondences $\mathbf{x}_{1j}, \mathbf{x}_{2j}, \dots, \mathbf{x}_{mj}$ different sets of subspaces U_1, U_2, \dots, U_m can be passed such that $\mathbf{x}_{ij} \in U_i$. Each choice of subspaces U_1, \dots, U_m gives a linear equation (6) on the elements of the Grassmann tensor. The Grassmann tensor can be obtained as the null vector of the matrix of coefficients of the resulting set of linear equations⁵.

⁵In Sect. 3.1 we prove the Grassmann tensor is unique, meaning that the matrix of coefficients of these linear equations has a 1D null space.

Lemma 3. *Consider a set of projection matrices $\mathbf{P}_1, \dots, \mathbf{P}_m$ with $\mathbf{P}_i \in \mathbb{R}^{s_i \times r}$ and $\mathbf{P}_i \neq \mathbf{0}$ for all i . Assume that there exists a valid profile $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ with $\alpha_k = 0$ such that $\mathcal{G}_{\alpha}(\mathbf{P}_1, \dots, \mathbf{P}_m)$ is nonzero. Then there exists a valid profile $\alpha' = (\alpha'_1, \alpha'_2, \dots, \alpha'_m)$ with $\alpha'_k > 0$ such that $\mathcal{G}_{\alpha'}(\mathbf{P}_1, \dots, \mathbf{P}_m)$ is nonzero.*

The lemma is proved by demonstrating that one row of a full-rank submatrix chosen according to α can be replaced with one nonzero row of \mathbf{P}_k such that the resulting submatrix remains full-rank. The full proof comes in the Supplementary Material.

3. Projective Reconstruction

Here, we state one version of the projective reconstruction theorem, proving the projective equivalence of two configurations $(\{\mathbf{P}_i\}, \{\mathbf{X}_j\})$ and $(\{\hat{\mathbf{P}}_i\}, \{\hat{\mathbf{X}}_j\})$ projecting into the same image points, given conditions on $(\{\hat{\mathbf{P}}_i\}, \{\hat{\mathbf{X}}_j\})$. In the next section, based on this theorem, we present an alternative theorem with conditions on the projective depths $\hat{\lambda}_{ij}$.

Theorem 2 (Projective Reconstruction). *Consider a configuration of m projection matrices and n points $(\{\mathbf{P}_i\}, \{\mathbf{X}_j\})$ where the matrices $\mathbf{P}_i \in \mathbb{R}^{s_i \times r}$ are generic, $\sum_{i=1}^m (s_i - 1) \geq r$, and $s_i \geq 3$ for all views⁶, and the points $\mathbf{X}_j \in \mathbb{R}^r$ are sufficiently many and in general position. Given a second configuration $(\{\hat{\mathbf{P}}_i\}, \{\hat{\mathbf{X}}_j\})$ that satisfies*

$$\hat{\mathbf{P}}_i \hat{\mathbf{X}}_j = \hat{\lambda}_{ij} \mathbf{P}_i \mathbf{X}_j \quad (7)$$

for some scalars $\{\hat{\lambda}_{ij}\}$, if

(C1) $\hat{\mathbf{X}}_j \neq \mathbf{0}$ for all j , and

(C2) $\hat{\mathbf{P}}_i \neq \mathbf{0}$ for all i , and

(C3) there exists a non-singular $r \times r$ submatrix $\hat{\mathbf{Q}}$ of $\hat{\mathbf{P}} = \text{stack}(\hat{\mathbf{P}}_1, \hat{\mathbf{P}}_2, \dots, \hat{\mathbf{P}}_m)$ containing strictly fewer than s_i rows from each \mathbf{P}_i . (equivalently $\mathcal{G}_{\alpha}(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_m) \neq \mathbf{0}$ for some valid profile α),

then the two configurations $(\{\mathbf{P}_i\}, \{\mathbf{X}_j\})$ and $(\{\hat{\mathbf{P}}_i\}, \{\hat{\mathbf{X}}_j\})$ are projectively equivalent.

It is important to observe the theorem does not assume a priori that the projective depths $\hat{\lambda}_{ij}$ are nonzero. At a first glance, this theorem might seem to be of no use, especially because condition (C3) looks hard to verify for a given setup $\{\hat{\mathbf{P}}_i\}$. But, this theorem is important as it forms the basis of our theory,

⁶We could have assumed the milder condition of $s_i \geq 3$ for at least one i . Our assumption avoids unnecessary complications.

by giving the minimal required conditions on the setup $(\{\hat{\mathbf{P}}_i\}, \{\hat{\mathbf{X}}_j\})$, from which simpler necessary conditions can be obtained.

Overview of the proof of Theorem 2 is as follows. Given the profile $\alpha = (\alpha_1, \dots, \alpha_m)$ from condition (C3),

1. for the special case of $\alpha_i \geq 1$ for all i , we prove that the Grassmann tensors $\mathcal{G}_\alpha(\mathbf{P}_1, \dots, \mathbf{P}_m)$ and $\mathcal{G}_\alpha(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_m)$ are equal up to a scaling factor, (Sect. 3.1).
2. Using the theory of Hartley and Schaffalitzky [3], we show that $(\{\mathbf{P}_i\}, \{\mathbf{X}_j\})$ and $(\{\hat{\mathbf{P}}_i\}, \{\hat{\mathbf{X}}_j\})$ are projectively equivalent for the special case of $\alpha_i \geq 1$ for all i , (Sect. 3.2).
3. We prove the theorem for the general case where some of α_i -s might be zero, and hence the number of views can be arbitrarily large, (Sect. 3.3).

3.1. The uniqueness of the Grassmann tensor

The main purpose of this subsection is to show that if $\hat{\mathbf{X}}_j \neq \mathbf{0}$ for all j , the relations $\hat{\mathbf{P}}_i \hat{\mathbf{X}}_j = \hat{\lambda}_{ij} \mathbf{P}_i \mathbf{X}_j$ imply that the Grassmann tensor $\mathcal{G}_\alpha(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_m)$ is equal to $\mathcal{G}_\alpha(\mathbf{P}_1, \dots, \mathbf{P}_m)$ up to a scaling factor. This implies that the Grassmann tensor is unique up to scale given a set of image points \mathbf{x}_{ij} obtained from $\lambda_{ij} \mathbf{x}_{ij} = \mathbf{P}_i \mathbf{X}_j$.

Theorem 3. *Consider a setup $(\{\mathbf{P}_i\}, \{\mathbf{X}_j\})$ of m generic projection matrices and n points, in general position and sufficiently many, and a valid profile $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ such that $\alpha_i \geq 1$ for all i . Now, for any other configuration $(\{\hat{\mathbf{P}}_i\}, \{\hat{\mathbf{X}}_j\})$ with $\hat{\mathbf{X}}_j \neq \mathbf{0}$ for all j , the set of relations*

$$\hat{\mathbf{P}}_i \hat{\mathbf{X}}_j = \hat{\lambda}_{ij} \mathbf{P}_i \mathbf{X}_j \quad (8)$$

implies $\mathcal{G}_\alpha(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_m) = \beta \mathcal{G}_\alpha(\mathbf{P}_1, \dots, \mathbf{P}_m)$ for some scalar β .

Due to lack of space, here we give the idea of the proof and present a formal proof in the Supplementary Material.

We consider two submatrices \mathbf{Q} and \mathbf{Q}' of $\mathbf{P} = \text{stack}(\mathbf{P}_1, \dots, \mathbf{P}_m)$ chosen according to the valid profile $\alpha = (\alpha_1, \dots, \alpha_m)$, such that all rows of \mathbf{Q} and \mathbf{Q}' are equal except for the l -th rows \mathbf{q}_l^T and \mathbf{q}'_l^T , which are chosen from different rows of \mathbf{P}_k . We also represent by $\hat{\mathbf{Q}}$ and $\hat{\mathbf{Q}}'$ the corresponding submatrices of $\hat{\mathbf{P}} = \text{stack}(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_m)$. Then we show that if $\det(\mathbf{Q}) \neq 0$, the equations (8) imply

$$\det(\hat{\mathbf{Q}}') = \frac{\det(\mathbf{Q}')}{\det(\mathbf{Q})} \det(\hat{\mathbf{Q}}). \quad (9)$$

The rest of the proof is as follows: By starting with a submatrix \mathbf{Q} of \mathbf{P} according to α , and iteratively updating \mathbf{Q} by changing one row at a time in the way described above, we can finally traverse all possible submatrices chosen according to α . Due to genericity we assume that all submatrices of \mathbf{P} chosen according to α are non-singular⁷. Therefore, (9) implies that during the traversal procedure the ratio $\beta = \det(\hat{\mathbf{Q}})/\det(\mathbf{Q})$ stays the same. This means that each element of $\mathcal{G}_\alpha(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_m)$ is β times the corresponding element of $\mathcal{G}_\alpha(\mathbf{P}_1, \dots, \mathbf{P}_m)$, implying $\mathcal{G}_\alpha(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_m) = \beta \mathcal{G}_\alpha(\mathbf{P}_1, \dots, \mathbf{P}_m)$.

The relation (9) is obtained in two steps. The first step is to write equations (8) in matrix form as

$$\mathcal{M}(\mathbf{X}_j) \begin{pmatrix} \hat{\lambda}_j \\ \hat{\mathbf{X}}_j \end{pmatrix} = \mathbf{0}, \quad j = 1, 2, \dots, n, \quad (10)$$

where $\hat{\lambda}_j = [\hat{\lambda}_{1j}, \dots, \hat{\lambda}_{mj}]^T$, and

$$\mathcal{M}(\mathbf{X}) = \begin{bmatrix} \mathbf{P}_1 \mathbf{X} & & & \hat{\mathbf{P}}_1 \\ & \mathbf{P}_2 \mathbf{X} & & \hat{\mathbf{P}}_2 \\ & & \ddots & \vdots \\ & & & \mathbf{P}_m \mathbf{X} & \hat{\mathbf{P}}_m \end{bmatrix}. \quad (11)$$

The matrix $\mathcal{M}(\mathbf{X})$ is $(\sum_i s_i) \times (m+r)$, and therefore a tall matrix. Due to the assumption $\hat{\mathbf{X}}_j \neq \mathbf{0}$ in Theorem 3, we conclude that $\mathcal{M}(\mathbf{X}_j)$ is rank deficient for all \mathbf{X}_j . Then, considering the fact that $\mathcal{M}(\mathbf{X})$ is rank deficient for sufficiently many points \mathbf{X}_j in general position, we show that $\mathcal{M}(\mathbf{X})$ is rank deficient for all $\mathbf{X} \in \mathbb{R}^r$. Therefore, for all $(m+r) \times (m+r)$ submatrices $\mathcal{M}'(\mathbf{X})$ of $\mathcal{M}(\mathbf{X})$ we have $\det(\mathcal{M}'(\mathbf{X})) = 0$.

The second step is to choose a proper value for \mathbf{X} and a proper submatrix $\mathcal{M}'(\mathbf{X})$ of $\mathcal{M}(\mathbf{X})$, such that (9) follows from $\det(\mathcal{M}'(\mathbf{X})) = 0$. This proper value for \mathbf{X} is $\mathbf{Q}^{-1} \mathbf{e}_l$, where \mathbf{e}_l is the l -th standard basis and l is the row which is different in \mathbf{Q} and \mathbf{Q}' , as defined above. The submatrix $\mathcal{M}'(\mathbf{X})$, is made by choosing the corresponding rows of $\mathbf{P} = \text{stack}(\mathbf{P}_1, \dots, \mathbf{P}_m)$ contributing to making \mathbf{Q} , choosing the corresponding row \mathbf{q}_l^T of \mathbf{P}_k contributing to making \mathbf{Q}' , and choosing one extra row from each \mathbf{P}_i for $i \neq k$. The details are given in the Supplementary Material.

3.2. Proof of Theorem 2 for the special case of $\alpha_i \geq 1$

Lemma 4. *Theorem 2 is true for the special case of $\alpha_i \geq 1$ for all i .*

The steps of the proof are: Given α from condition (C3) of Theorem 2, Theorem 3 tells $\mathcal{G}_\alpha(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_m) =$

⁷though the proof is possible under a slightly milder assumption,

$\beta \mathcal{G}_\alpha(\mathbf{P}_1, \dots, \mathbf{P}_m)$. From (C3) it follows that $\beta \neq 0$. Thus, Theorem 1 (proved in [3]), suggests that $\{\mathbf{P}_i\}$ and $\{\hat{\mathbf{P}}_i\}$ are projectively equivalent. Then, using the Triangulation Lemma 2, we prove that $(\{\mathbf{P}_i\}, \{\mathbf{X}_j\})$ and $(\{\hat{\mathbf{P}}_i\}, \{\hat{\mathbf{X}}_j\})$ are projectively equivalent. The formal proof is given in the Supplementary Material.

3.3. The general case

To prove Theorem 2 in the general case, where we might have $\alpha_i = 0$ for some elements of the valid profile $\alpha = (\alpha_1, \dots, \alpha_m)$, given in condition (C3) of the theorem, we proceed as follows: By (C3) we have $\mathcal{G}_\alpha(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_m) \neq \mathbf{0}$, by Lemma 3, for each view k , there exists a valid profile $\alpha^{(k)}$ for which $\alpha_k^{(k)} \geq 1$ and the Grassmann tensor $\mathcal{G}_{\alpha^{(k)}}(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_m)$ is nonzero. Define $I_k = \{i \mid \alpha_i^{(k)} \geq 1\}$. Lemma (4) proves for each I_k that the configurations $(\{\mathbf{P}_i\}_{I_k}, \{\mathbf{X}_j\})$ and $(\{\hat{\mathbf{P}}_i\}_{I_k}, \{\hat{\mathbf{X}}_j\})$ are projectively equivalent. As $\cup_k I_k = \{1, \dots, m\}$, using Lemma 1 we show the projective equivalence holds for the whole set of views, that is $(\{\mathbf{P}_i\}, \{\mathbf{X}_j\})$ and $(\{\hat{\mathbf{P}}_i\}, \{\hat{\mathbf{X}}_j\})$. See the Supplementary Material for the complete proof.

4. Restricting projective depths

This section provides a second version of Theorem 2 in which it is assumed that $\hat{\lambda}_{ij}$ -s are all nonzero, instead of putting restrictions on $(\{\hat{\mathbf{P}}_i\}, \{\hat{\mathbf{X}}_j\})$.

Theorem 4 (Projective Reconstruction). *Consider a configuration of m projection matrices and n points $(\{\mathbf{P}_i\}, \{\mathbf{X}_j\})$ where the matrices $\mathbf{P}_i \in \mathbb{R}^{s_i \times r}$ are generic and as many such that $\sum_{i=1}^m (s_i - 1) \geq r$, and $s_i \geq 3$ for all views, and the points $\mathbf{X}_j \in \mathbb{R}^r$ are sufficiently many and in general position. Now, for any second configuration $(\{\hat{\mathbf{P}}_i\}, \{\hat{\mathbf{X}}_j\})$ satisfying*

$$\hat{\mathbf{P}}_i \hat{\mathbf{X}}_j = \hat{\lambda}_{ij} \mathbf{P}_i \mathbf{X}_j. \quad (12)$$

for some nonzero scalars $\hat{\lambda}_{ij} \neq 0$, the configuration $(\{\hat{\mathbf{P}}_i\}, \{\hat{\mathbf{X}}_j\})$ is projectively equivalent to $(\{\mathbf{P}_i\}, \{\mathbf{X}_j\})$.

The condition $\hat{\lambda}_{ij} \neq 0$ is not tight, and used here to avoid complexity. In Sect. 5 we will discuss that the theorem is provable under milder restrictions. However, by proving projective equivalence, it eventually follows that all $\hat{\lambda}_{ij}$ -s are nonzero. We prove the theorem after giving required lemmas.

Lemma 5. *Consider m matrices $\hat{\mathbf{P}}_1, \hat{\mathbf{P}}_2, \dots, \hat{\mathbf{P}}_m$ with $\hat{\mathbf{P}}_i \in \mathbb{R}^{s_i \times r}$, for which a valid profile can be defined. Also, assume that $\hat{\mathbf{P}} = \text{stack}(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_m)$ has full column rank. If $\hat{\mathbf{P}}$ has no non-singular $r \times r$ submatrix chosen by strictly fewer than s_i rows from each $\hat{\mathbf{P}}_i$,*

then there exists a nonempty subset of views $I \neq \emptyset$ with $\sum_{i \in I} s_i \leq r$, such that $\hat{\mathbf{P}}^I = \text{stack}(\{\hat{\mathbf{P}}_k\}_{k \in I})$ has rank $r' = r - \sum_{i \in I} s_i$. Further, the row space of $\hat{\mathbf{P}}^I$ is spanned by the rows of an $r' \times r$ submatrix $\hat{\mathbf{Q}}^I = \text{stack}(\{\hat{\mathbf{Q}}_k\}_{k \in I})$ of $\hat{\mathbf{P}}^I$, where each $\hat{\mathbf{Q}}_k$ is created by choosing strictly less than s_i rows from $\hat{\mathbf{P}}_k$.

The proof is given in the Supplementary Material.

Lemma 6. *Under the conditions of Theorem 4, if the matrix $\hat{\mathbf{P}} = \text{stack}(\hat{\mathbf{P}}_1, \hat{\mathbf{P}}_2, \dots, \hat{\mathbf{P}}_m)$ has full column rank, it has a non-singular $r \times r$ submatrix chosen according to some valid profile $\alpha = (\alpha_1, \dots, \alpha_m)$.*

Proof. To get a contradiction, assume that $\hat{\mathbf{P}}$ does not have any full-rank submatrix with any valid profile. Then by Lemma 5, there exists a nonempty index set $I \neq \emptyset$ with $\sum_{i \in I} s_i \leq r$, and $\hat{\mathbf{P}}^I = \text{stack}(\{\hat{\mathbf{P}}_k\}_{k \in I})$ has a row space of dimension $r' = r - \sum_{i \in I} s_i < r$, spanned by the rows of an $r' \times r$ matrix $\hat{\mathbf{Q}}^I = \text{stack}(\{\hat{\mathbf{Q}}_k\}_{k \in I})$, where each $\hat{\mathbf{Q}}_k$ consists of strictly less than s_k rows of $\hat{\mathbf{P}}_k$. Notice that some $\hat{\mathbf{Q}}_k$ -s might have zero rows. By relabeling the views if necessary, we assume that $\hat{\mathbf{P}}^I = \text{stack}(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_l)$ and $\hat{\mathbf{Q}}^I = \text{stack}(\hat{\mathbf{Q}}_1, \dots, \hat{\mathbf{Q}}_l)$ (thus $I = \{l+1, \dots, m\}$). As rows of $\hat{\mathbf{Q}}^I$ span the row space of $\hat{\mathbf{P}}^I$, we have $\hat{\mathbf{P}}^I = \mathbf{A} \hat{\mathbf{Q}}^I$ for some $(\sum_{i=1}^l s_i) \times r'$ matrix \mathbf{A} . From (12), we have $\hat{\mathbf{P}}_i \hat{\mathbf{X}}_j = \hat{\lambda}_{ij} \mathbf{P}_i \mathbf{X}_j$ and, as a result, $\hat{\mathbf{Q}}_i \hat{\mathbf{X}}_j = \hat{\lambda}_{ij} \mathbf{Q}_i \mathbf{X}_j$, where \mathbf{Q}_i is the submatrix of \mathbf{P}_i corresponding to $\hat{\mathbf{Q}}_i$. This gives

$$\hat{\mathbf{P}}^I \hat{\mathbf{X}}_j = \text{diag}(\mathbf{P}_1 \mathbf{X}_j, \mathbf{P}_2 \mathbf{X}_j, \dots, \mathbf{P}_l \mathbf{X}_j) \hat{\lambda}_j \quad (13)$$

$$\hat{\mathbf{Q}}^I \hat{\mathbf{X}}_j = \text{diag}(\mathbf{Q}_1 \mathbf{X}_j, \mathbf{Q}_2 \mathbf{X}_j, \dots, \mathbf{Q}_l \mathbf{X}_j) \hat{\lambda}_j \quad (14)$$

where $\text{diag}(\cdot)$ makes a block diagonal matrix out of its arguments, and $\hat{\lambda}_j = [\hat{\lambda}_{1j}, \dots, \hat{\lambda}_{lj}]^T$. From $\hat{\mathbf{P}}^I = \mathbf{A} \hat{\mathbf{Q}}^I$, then we have $\mathcal{M}(\mathbf{X}_j) \hat{\lambda}_j = \mathbf{0}$, where

$$\mathcal{M}(\mathbf{X}) = \text{diag}(\mathbf{P}_1 \mathbf{X}, \dots, \mathbf{P}_l \mathbf{X}) - \mathbf{A} \text{diag}(\mathbf{Q}_1 \mathbf{X}, \dots, \mathbf{Q}_l \mathbf{X}).$$

Notice that, $\mathcal{M}(\mathbf{X})$ is $(\sum_{i=1}^l s_i) \times l$, and thus a tall matrix. As $\hat{\lambda}_j \neq \mathbf{0}$ (since $\hat{\lambda}_{ij} \neq 0$ for all i, j), $\mathcal{M}(\mathbf{X}_j) \hat{\lambda}_j = \mathbf{0}$ implies that $\mathcal{M}(\mathbf{X}_j)$ is rank deficient. Since, $\mathcal{M}(\mathbf{X})$ is rank-deficient for sufficiently many points \mathbf{X}_j in general position, with the same argument as given in the proof of Theorem 3 (See the Supplementary Material), we conclude that $\mathcal{M}(\mathbf{X})$ is rank-deficient for all $\mathbf{X} \in \mathbb{R}^r$. Notice that $\hat{\mathbf{Q}}^I$ is $r' \times r$ with $r' < r$. As \mathbf{P}_i -s are generic, we can take a nonzero vector \mathbf{Y} in the null space of $\hat{\mathbf{Q}}^I$ such that no matrix $\hat{\mathbf{P}}_i$ for $i = 1, \dots, l$ has \mathbf{Y} in its null space⁸. In this case, we have $\mathbf{Q}_i \mathbf{Y} = \mathbf{0}$ for all i , implying $\mathcal{M}(\mathbf{Y}) = \text{diag}(\mathbf{P}_1 \mathbf{Y}, \dots, \mathbf{P}_l \mathbf{Y})$. Now, from

⁸ \mathbf{Y} must be chosen from $\mathcal{N}(\hat{\mathbf{Q}}^I) \setminus \cup_{i=1}^l (\mathcal{N}(\hat{\mathbf{Q}}^I) \cap \mathcal{N}(\hat{\mathbf{P}}_i))$ which is nonempty (in fact open and dense in $\mathcal{N}(\hat{\mathbf{Q}}^I)$) for generic \mathbf{P}_i -s.

$\mathbf{Y} \notin \mathcal{N}(\hat{\mathbf{P}}_i)$, we have $\mathbf{P}_i \mathbf{Y} \neq 0$ for $i = 1, \dots, l$. This implies that $\mathcal{M}(\mathbf{Y}) = \text{diag}(\mathbf{P}_1 \mathbf{Y}, \dots, \mathbf{P}_l \mathbf{Y})$ has full column rank, contradicting the fact that $\mathcal{M}(\mathbf{X})$ is rank deficient for all $\mathcal{M}(\mathbf{X})$. \square

Proof of Theorem 4. Using Theorem 2 we just need to prove that the condition $\hat{\lambda}_{ij} \neq 0$ imply conditions (C1-C3) of Theorem 2. Assume that $\hat{\lambda}_{ij} \neq 0$ for some i and j , then from the genericity of \mathbf{P}_i and \mathbf{X}_j we have $\hat{\mathbf{P}}_i \hat{\mathbf{X}}_j = \hat{\lambda}_{ij} \mathbf{P}_i \mathbf{X}_j \neq 0$, implying $\hat{\mathbf{P}}_i \neq 0$ and $\hat{\mathbf{X}}_j \neq 0$. This means that $\hat{\lambda}_{ij} \neq 0$ for all i and j imply (C1) and (C2). Now, it is left to show that $\hat{\lambda}_{ij} \neq 0$ imply (C3), that is $\hat{\mathbf{P}}$ has a full-rank $r \times r$ submatrix chosen according to some valid profile α . This is proved in Lemma 6 for when $\hat{\mathbf{P}} = \text{stack}(\hat{\mathbf{P}}_1, \hat{\mathbf{P}}_2, \dots, \hat{\mathbf{P}}_m)$ has full column rank. We complete the proof by showing that $\hat{\mathbf{P}}$ always has full column rank.

Assume, $\hat{\mathbf{P}}$ is rank deficient. Consider the matrix $\hat{\mathbf{X}} = [\hat{\mathbf{X}}_1, \dots, \hat{\mathbf{X}}_m]$. The matrix $\hat{\mathbf{P}}\hat{\mathbf{X}}$ can always be factorized as $\hat{\mathbf{P}}\hat{\mathbf{X}} = \hat{\mathbf{P}}'\hat{\mathbf{X}}'$, with $\hat{\mathbf{P}}'$ and $\hat{\mathbf{X}}'$ respectively of the same dimensions as $\hat{\mathbf{P}}$ and $\hat{\mathbf{X}}$, such that $\hat{\mathbf{P}}'$ has full column rank. By defining the same block structure as $\hat{\mathbf{P}}$ and $\hat{\mathbf{X}}$ for $\hat{\mathbf{P}}'$ and $\hat{\mathbf{X}}'$, that is $\hat{\mathbf{P}} = \text{stack}(\hat{\mathbf{P}}'_1, \dots, \hat{\mathbf{P}}'_m)$ and $\hat{\mathbf{X}}' = [\hat{\mathbf{X}}'_1, \dots, \hat{\mathbf{X}}'_m]$, we observe that $\hat{\mathbf{P}}'_i \hat{\mathbf{X}}'_j = \hat{\mathbf{P}}_i \hat{\mathbf{X}}_j = \hat{\lambda}_{ij} \mathbf{P}_i \mathbf{X}_j$. As $\hat{\mathbf{P}}'$ has full column rank, from the discussion of the first half of the proof, we can say that $(\{\hat{\mathbf{P}}'_i\}, \{\hat{\mathbf{X}}'_j\})$ is projectively equivalent to $(\{\mathbf{P}_i\}, \{\mathbf{X}_j\})$. This implies that $\hat{\mathbf{X}}' = [\hat{\mathbf{X}}'_1, \dots, \hat{\mathbf{X}}'_m]$ has full row rank. As $\hat{\mathbf{P}}'$ and $\hat{\mathbf{X}}'$ both have maximum rank r , their $\hat{\mathbf{P}}'\hat{\mathbf{X}}' = \hat{\mathbf{P}}\hat{\mathbf{X}}$ has rank r , requiring $\hat{\mathbf{P}}$ to have full column rank, a contradiction. \square

5. Wrong solutions to projective factorization

Let us write equations $\hat{\lambda}_{ij} \mathbf{x}_{ij} = \hat{\mathbf{P}}_i \hat{\mathbf{X}}_j$ in matrix form

$$\hat{\Lambda} \odot [\mathbf{x}_{ij}] = \hat{\mathbf{P}} \hat{\mathbf{X}}, \quad (15)$$

where $\hat{\Lambda} \odot [\mathbf{x}_{ij}] = [\hat{\lambda}_{ij} \mathbf{x}_{ij}]$, $\hat{\mathbf{P}} = \text{stack}(\hat{\mathbf{P}}_1, \dots, \hat{\mathbf{P}}_m)$ and $\hat{\mathbf{X}} = [\hat{\mathbf{X}}_1, \dots, \hat{\mathbf{X}}_m]$. The factorization-based algorithms seek to find $\hat{\Lambda}$ such that $\hat{\Lambda} \odot [\mathbf{x}_{ij}]$ can be factorized as the product of a $(\sum_i s_i) \times r$ matrix $\hat{\mathbf{P}}$ by an $r \times n$ matrix $\hat{\mathbf{X}}$. If \mathbf{x}_{ij} -s are obtained from a set of projection matrices \mathbf{P}_i and points \mathbf{X}_j , according to $\mathbf{x}_{ij} = \mathbf{P}_i \mathbf{X}_j / \lambda_{ij}$, our theory says that any solution $(\hat{\Lambda}, \hat{\mathbf{P}}, \hat{\mathbf{X}})$ to (15), is equivalent to the true solution $(\Lambda, \mathbf{P}, \mathbf{X})$, if $(\hat{\Lambda}, \hat{\mathbf{P}}, \hat{\mathbf{X}})$ satisfies some special restrictions, such as conditions (C1-C3) on $\hat{\mathbf{P}}$ and $\hat{\mathbf{X}}$ in Theorem 2, or $\hat{\Lambda}$ having no zero element in Theorem 4. It is worth to see what degenerate (projectively nonequivalent) forms a solution $(\hat{\Lambda}, \hat{\mathbf{P}}, \hat{\mathbf{X}})$ to (15) can get when such restrictions are not completely imposed. This is important in the factorization-based

methods, in which sometimes such restrictions cannot be completely implemented.

The reader can check that Theorem 4 is provable under weaker assumptions than $\hat{\lambda}_{ij} \neq 0$, as follows

- (D1) The matrix $\hat{\Lambda} = [\hat{\lambda}_{ij}]$ has no zero rows,
- (D2) The matrix $\hat{\Lambda} = [\hat{\lambda}_{ij}]$ has no zero columns,
- (D3) For every subset I of views with $\sum_{i \in I} s_i < r$, the matrix $\hat{\Lambda}^{\bar{I}}$ has *sufficiently many* nonzero columns, where \bar{I} is the complement of I and $\hat{\Lambda}^{\bar{I}}$ is the matrix of $\hat{\Lambda}$ created by selecting rows according to \bar{I} .

Notice that (D1) and (D2), respectively guarantee (C1) and (C2) in Theorem 2. The condition (D3), guarantees that $\hat{\lambda}_j$ used in the proof of Lemma 6, is nonzero for sufficiently many j -s. Therefore, (D3) is used to guarantee (C3) in Theorem 2, that is $\hat{\mathbf{P}}$ has a nonzero minor chosen according to some valid profile.

It is easy to see how violating (D1) and (D2) can lead to a false solution to (15) (for example set $\hat{\mathbf{P}}_k = 0$ and $\hat{\lambda}_{kj} = 0$ for all j). In what comes next, we assume that (D1) and (D2) hold, and look for nontrivial false solutions to (15). From our discussion we can conclude that if $\hat{\Lambda}$ has no zero rows and no zero columns, false solutions to (15) are those in which all $r \times r$ submatrices of $\hat{\mathbf{P}}$ chosen with fewer than s_i rows from each $\hat{\mathbf{P}}_i$ are singular. In this case, if $\hat{\mathbf{P}}\hat{\mathbf{X}}$ is factorized such that $\hat{\mathbf{P}}$ has full column rank, according to Lemma 5, there exists a nonempty set of views I with $\sum_{i \in I} s_i \leq r$, according to which the matrix $\hat{\mathbf{P}}$ can be split into two submatrices \mathbf{P}^I and $\mathbf{P}^{\bar{I}}$, such that \mathbf{P}^I is of rank $r' = r - \sum_{i \in I} s_i$. We give an example showing how such a case can happen.

For a setup $(\{\mathbf{P}_i\}, \{\mathbf{X}_j\})$, partition the views into two subsets I and \bar{I} , such that $\sum_{i \in I} s_i \leq r$. Split \mathbf{P} into two submatrices $\mathbf{P}^I = \text{stack}(\{\mathbf{P}_i\}_{i \in I})$ and $\mathbf{P}^{\bar{I}} = \text{stack}(\{\mathbf{P}_i\}_{i \in \bar{I}})$. By possibly relabeling the views, we assume that $\mathbf{P} = \text{stack}(\mathbf{P}^I, \mathbf{P}^{\bar{I}})$. Notice that \mathbf{P}^I has $\sum_{i \in I} s_i$ rows, and therefore, at least an $r' = r - \sum_{i \in I} s_i$ dimensional null space. Consider the $r \times r'$ full-column-rank matrix \mathbf{N} whose columns are in the null space of \mathbf{P}^I . Also, let \mathbf{R} be the orthogonal projection matrix into the row space of \mathbf{P}^I . Divide the matrix $\mathbf{X} = [\mathbf{X}_1, \dots, \mathbf{X}_m]$ into two parts $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2]$ where $\mathbf{X}_1 = [\mathbf{X}_1, \dots, \mathbf{X}_{r'}]$ and $\mathbf{X}_2 = [\mathbf{X}_{r'+1}, \dots, \mathbf{X}_m]$. Define the corresponding submatrices of $\hat{\mathbf{P}}$ and $\hat{\mathbf{X}}$ as

$$\hat{\mathbf{P}}^I = \mathbf{P}^I, \quad \hat{\mathbf{P}}^{\bar{I}} = \mathbf{P}^{\bar{I}} \mathbf{X}_1 \mathbf{N}^T, \quad (16)$$

$$\hat{\mathbf{X}}_1 = \mathbf{R} \mathbf{X}_1 + \mathbf{N}, \quad \hat{\mathbf{X}}_2 = \mathbf{R} \mathbf{X}_2. \quad (17)$$

One can easily check that

$$\hat{\mathbf{P}}\hat{\mathbf{X}} = \begin{bmatrix} \hat{\mathbf{P}}^I \\ \hat{\mathbf{P}}^{\bar{I}} \end{bmatrix} [\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2] = \begin{bmatrix} \mathbf{P}^I \mathbf{X}_1 & \mathbf{P}^I \mathbf{X}_2 \\ \mathbf{P}^{\bar{I}} \mathbf{X}_1 & 0 \end{bmatrix} = \hat{\Lambda} \odot (\mathbf{P}\mathbf{X}) \quad (18)$$

where $\hat{\Lambda}$ has a block structure of the form

$$\hat{\Lambda} = \begin{bmatrix} \hat{\Lambda}^I \\ \hat{\Lambda}^{\bar{I}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad (19)$$

From (16) it is clear that the row space of $\hat{P}^{\bar{I}}$ is spanned by columns of \mathbf{N} , which makes an r' dimensional space⁹ (compare Lemma 5). One can check that $\hat{P} = \text{stack}(\hat{P}^I, \hat{P}^{\bar{I}})$ has no non-singular submatrix chosen by less than s_i rows from each \hat{P}_i . Also, notice that $\hat{\Lambda}^{\bar{I}}$, the lower block of $\hat{\Lambda}$, has only r' nonzero columns, no matter how large the number of columns n is. This is how (D3) is violated.

Using the above style for finding the wrong solutions, $\hat{\Lambda}^I$ can have at most r' nonzero columns. Unfortunately, unlike the case of $\mathbb{P}^3 \rightarrow \mathbb{P}^2$ projections, this is not always the case. In other words, *sufficiently many* in the condition (D3) to rule out false solutions does not mean more than $r' = r - \sum_{i \in I} s_i$. Instead, the limit for the number of nonzero columns allowable in a wrong solution is as many such that the rank of $\hat{\Lambda}^I \odot (P^I X)$ is not more than r' . This is necessary for having a wrong solution as $\hat{\Lambda}^I \odot (P^I X) = \hat{P}^I \hat{X}$, and \hat{P}^I cannot have a rank of more than r' according to Lemma 5. One can also show that this is a sufficient condition for having a wrong solution. For the classic case of $\mathbb{P}^3 \rightarrow \mathbb{P}^2$ projections, the index set I can only have one member and $r' = r - \sum_{i \in I} s_i = 4 - 3 = 1$. The condition $\mathcal{R}\text{ank}(\hat{\Lambda}^I \odot (P^I X)) \leq r' = 1$, implies that only one column of $\hat{\Lambda}^I$ can be nonzero, causing $\hat{\Lambda}$ to have a cross-shaped structure. Therefore, the theory given in [7] follows as a special case of ours.

6. Conclusion

This paper investigates projective reconstruction for arbitrary dimensional projections. We obtain the following results for a generic setup:

- The Grassmann tensor obtained from the image points \mathbf{x}_{ij} is unique (up to a scaling factor).
- Any solution to the set of equations $\hat{\lambda}_{ij} \mathbf{x}_{ij} = \hat{P}_i \hat{X}_j$ is projectively equivalent to the true setup, if the \hat{P}_i -s and \hat{X}_j -s are nonzero and $\hat{P} = \text{stack}(\hat{P}_1, \dots, \hat{P}_m)$ has a non-singular $r \times r$ submatrix created by choosing strictly fewer than s_i rows from each $\hat{P}_i \in \mathbb{R}^{s_i \times r}$.
- Any solution to the set of equations $\hat{\lambda}_{ij} \mathbf{x}_{ij} = \hat{P}_i \hat{X}_j$ is projectively equivalent to the true setup if $\hat{\lambda}_{ij} \neq 0$ for all i, j .

⁹In our example, the row space of $\hat{P}^{\bar{I}}$ is the null space of \hat{P}^I . This is not necessary for a wrong solution, and is chosen here to simplify the example.

- False solutions to the projective factorization problem $\hat{\Lambda} \odot [\mathbf{x}_{ij}] = \hat{P} \hat{X}$ in the general case can be much more complex than in the case of projections $\mathbb{P}^3 \rightarrow \mathbb{P}^2$.

A possible extension to this work is to consider the case of incomplete data, where some of the image points \mathbf{x}_{ij} are missing. It would be also useful to compile a simplified list of all the required generic properties needed for the proof of projective reconstruction. This is because, in almost all applications the projection matrices and points have a special structure, meaning they are members of a nongeneric set. It is now a nontrivial question whether the restriction of the genericity conditions to this nongeneric set is relatively generic.

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