# On state observers – Take 2

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**Abstract** This is the author's second attempt to provide a characterization for asymptotic functional state observers in the category of linear time-invariant finite-dimensional systems in input / state / output form in terms of a Sylvester-type matrix equation with a proof that only uses state space and transfer function methods. The characterizing equation was already proposed in Luenberger's original work on state observers, but to prove that it is not only sufficient but also necessary when the observed system has no stable uncontrollable modes turns out to be surprisingly hard. The crux of the problem is that in a classical observer interconnection both the output and the input of the observed system enter the observer and hence also the observer error system as separate inputs. They are not independent signals, though, since they are jointly constrained by the equations of the observed system. The first attempt by the author (see the list of references) contained a subtle error in the proof of the main result. To fix this error some new intermediate results are needed and the final proof is sufficiently different to warrant this paper. As a bonus, details on how to observe stable uncontrollable modes are also provided. The presentation is mostly self contained with only occasional references to standard results in linear system theory. It is an absolute pleasure to dedicate this paper to my friend and colleague Harry Trentelman on the occasion of his sixtieth birthday. Harry and I have worked together on linear system theory for the last five years and our behavioral internal model principle for observers (joined work with Jan Willems) provides an alternative proof for the result reported here.

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# 1 Problem formulation

Consider the linear time-invariant finite-dimensional system in state space form given by

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \\ z &= Vx, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $V \in \mathbb{R}^{r\times n}$.

We will be interested in the characterization of asymptotic observers for $z$ given $u$ and $y$. In particular, we will be interested in observers of the following type usually considered in the geometric control literature:

$$\begin{aligned} \dot{v} &= Kv + Ly + Mu, \\ \hat{z} &= Pv + Qy, \end{aligned} \tag{2}$$

where $K \in \mathbb{R}^{s\times s}$, $P \in \mathbb{R}^{r\times s}$ and the other matrices are real and appropriately sized. Note that $P$ can be rectangular (tall or wide) and/or not of full rank. The asymptotic condition for this type of observer is

$$\lim_{t\to\infty}[\hat{z}(t) - z(t)] = 0 \tag{3}$$

for every choice of input $u$ and initial conditions $x(0)$ and $v(0)$. We then say that system (2) is an *asymptotic observer* for system (1).

The problem considered in this paper is to characterize when a *given* observer of the form (2) is an asymptotic observer for a *given* observed system (1). See [1], in particular Section 3.1, for a detailed discussion of the relevant literature.

# 2 Problem reduction

In the observer *characterization* problem, both the observed system and the observer are given and fixed, so we can not modify them without changing the problem. We can, however, show results of the type Observer A is an asymptotic observer for System A if and only if Observer B is an asymptotic observer for System B, where both Observer A and B as well as System A and B are related by equations (one of the pairs may even be identical). This then allows to reduce the problem in the case where Observer B and/or System B are simpler than the A variety.

As a first result of this type we show that only the observable part of the observer (2) is relevant for the observer characterization problem. Consider the (dual) Kalman decomposition for the pair $(P,K)$: There exists an invertible $S \in \mathbb{R}^{s\times s}$ such that

$$SKS^{-1} = \begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix} \qquad \text{and} \qquad PS^{-1} = \begin{bmatrix} P_1 & 0 \end{bmatrix},$$

where the pair $(P_1, K_{11})$ is observable. Now split

$$SL = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \qquad SM = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \qquad \text{and} \qquad Sv = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

and consider the reduced observer

$$\begin{aligned} \dot{v}_1 &= K_{11} v_1 + L_1 y + M_1 u, \\ \hat{z} &= P_1 v_1 + Qy. \end{aligned} \tag{4}$$

Note that this observer is an observable system, i.e. $v_1$ is observable from $((u,y),\hat{z})$ in this observer. We call such observers *observable asymptotic observers*. We now have the following result, cf. [2, Proposition 3.69].

**Proposition 1.** *System* (2) *is an asymptotic observer for system* (1) *if and only if the reduced system* (4) *is an (observable) asymptotic observer for system* (1)*.*

*Proof.* The proof follows from the observation that, given $(u,y)$, system (4) started with $v_1(0)$ produces the same output as system (2) started with $v(0)$.

In order to simplify the notation, we will assume in the next section that $(P,K)$ itself is observable. We only need to replace $(K,L,M,P,Q)$ by $(K_{11},L_1,M_1,P_1,Q)$ in the resulting characterization to recover the general case.

In a second step, we can simplify the observed system by removing any uncontrollable stable modes. The corresponding linear functions of the state go to zero irrespective of the applied input $u$ and hence do not need to be observed at all. We can make this discussion more precise as follows. Consider the unstable/stable Kalman decomposition for the pair $(A,B)$: There exists an invertible $T \in \mathbb{R}^{n\times n}$ such that

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \qquad \text{and} \qquad TB = \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix},$$

where $(A_{11}, B_1)$ is controllable, $A_{22}$ is anti-Hurwitz and $A_{33}$ is Hurwitz. Now split

$$CT^{-1} = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}, \qquad VT^{-1} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}, \qquad \text{and} \qquad Tx = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

We now have the following result.

**Proposition 2.** *Let* $\dim(x_3) < \dim(x)$*. Then system* (2) *is an asymptotic observer for system* (1) *if and only if it is an asymptotic observer for the reduced system*

$$\begin{aligned}\begin{bmatrix}\dot{x}_{1r}\\ \dot{x}_{2r}\end{bmatrix} &= \begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix}\begin{bmatrix}x_{1r}\\ x_{2r}\end{bmatrix} + \begin{bmatrix}B_1\\ 0\end{bmatrix} u_r,\\ y_r &= \begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}x_{1r}\\ x_{2r}\end{bmatrix},\\ z_r &= \begin{bmatrix}V_1 & V_2\end{bmatrix}\begin{bmatrix}x_{1r}\\ x_{2r}\end{bmatrix}.\end{aligned} \tag{5}$$

*The latter is in the sense that* $u := u_r$ *and* $y := y_r$ *in the observer yields* $\lim_{t\to\infty}[\hat{z}(t) - z_r(t)] = 0$ *for all choices of* $x_{1r}(0)$, $x_{2r}(0)$, $v(0)$ *and* $u_r$.

The proof will use the following characterization of output stability, cf. [2, Proposition 3.50]. We prove a slightly extended version.

**Lemma 1.** *Consider the linear time-invariant finite-dimensional system in state space form given by*

$$\begin{aligned}\dot{x} &= Ax + Bu,\\ y &= Cx,\end{aligned}$$

*where* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ *and* $C \in \mathbb{R}^{p\times n}$. *Then* $\lim_{t\to\infty} y(t) = 0$ *for all choices of* $x(0)$ *and* $u$ *if and only if* $CR(A,B) = 0$ *and the co-restriction of* $A$ *to the quotient space* $\mathbb{R}^n/\mathfrak{N}(C,A)$ *is Hurwitz. Here,* $R(A,B) = \begin{bmatrix}B & AB & \dots & A^{n-1}B\end{bmatrix}$ *is the reachability matrix of the pair* $(A,B)$ *and* $\mathfrak{N}(C,A) \subset \mathbb{R}^n$ *is the unobservable subspace of the pair* $(C,A)$. *If in addition* $(C,A)$ *is observable then* $A$ *is Hurwitz and* $B = 0$.

*Proof.* Assume there exists $x_0 \in \mathfrak{R}(A,B) := \operatorname{Im} R(A,B)$, the reachable subspace of the pair $(A,B)$, with $Cx_0 \neq 0$. Since $x_0 \in \mathfrak{R}(A,B)$, there exists $u$ and a corresponding trajetory $x$ that oscillates between 0 and $x_0$, contradicting $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} Cx(t) = 0$. Hence $CR(A,B) = 0$. If $\mathfrak{N}(C,A) = \mathbb{R}^n$ there is nothing to prove for the co-restriction of $A$. Assume $\mathfrak{N}(C,A) \neq \mathbb{R}^n$ and assume that $A$ is not stable on $\mathbb{R}^n/\mathfrak{N}(C,A)$. Then there exists $0 \neq x_0 \in \mathbb{R}^n$ with $x_0 \notin \mathfrak{N}(C,A)$ and $Ax_0 = \lambda x_0$ for a $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$. It is $x_0 \notin \operatorname{Ker} C$ since the span of $x_0$ is $A$-invariant but $\mathscr{N}(C,A)$ is the largest $A$-invariant subspace of $\operatorname{Ker} C$. Choosing $x(0) = x_0$ and $u = 0$ yields a trajectory with $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} Cx(t) \neq 0$, a contradiction. Hence $A$ is stable on $\mathbb{R}^n/\mathfrak{N}(C,A)$.

Conversely let $CR(A,B) = 0$ and let $A$ be stable on $\mathbb{R}^n/\mathfrak{N}(C,A)$. Let $x(0) = x_0 \in \mathbb{R}^n$ and let $u$ be arbitrary. Then

$$y(t) = Cx(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau,$$

where the integral is an element of $\mathfrak{R}(A,B)$ and hence of $\operatorname{Ker} C$. Let $\mathbb{R}^n = \mathfrak{N}(C,A) \oplus \mathfrak{W}$ and decompose $x_0 = n_0 + w_0$ with $n_0 \in \mathfrak{N}(C,A)$ and $w_0 \in \mathfrak{W}$. Since $\mathfrak{N}(C,A)$ is $A$-invariant and contained in $\operatorname{Ker} C$ it follows that $y(t) = Ce^{At}w_0$. But $A$ is stable on $\mathfrak{W}$ and hence $\lim_{t\to\infty} y(t) = 0$.

If in addition $(C,A)$ is observable then $\mathfrak{N}(C,A)=\{0\}$ and $A$ is Hurwitz. Furthermore, $C\begin{bmatrix}B\ AB\ \ldots\ A^{n-1}B\end{bmatrix}=0$ implies

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} B = 0$$

and hence $B=0$ by observability.

*Proof (of Proposition 2).* Note that system (1) and the Kalman decomposed system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\ 0 & A_{22} & 0\\ 0 & 0 & A_{33}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}B_1\\ 0\\ 0\end{bmatrix}u,$$

$$y = \begin{bmatrix}C_1 & C_2 & C_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix},$$

$$z = \begin{bmatrix}V_1 & V_2 & V_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

have the same external behavior in terms of the variables $(u,y,z)$. In order to enable a direct comparison of the trajectories of system (1) and the reduced system (5), whenever we are given an initial condition $x(0)$ for system (1), we will use $x_{1r}(0) := x_1(0)$, $x_{2r}(0) := x_2(0)$ in the reduced system (5). Then, system (1) and the reduced system (5) have the same output, i.e. $y=y_r$ and $z=z_r$, if $x_3(0)=0$ and $u_r=u$.

Let system (2) be an asymptotic observer for system (1), then $\lim_{t\to\infty}[\hat{z}(t)-z(t)]=0$ for every choice of $x(0)$, $v(0)$ and $u$, in particular for those $x(0)$ with $x_3(0)=0$. But in that case $x_{1r}(0) := x_1(0)$, $x_{2r}(0) := x_2(0)$ and $u_r := u$ in system (5) implies $y=y_r$ and hence the observer output is the same for both observed systems. Moreover, $z=z_r$ in this case and hence $\lim_{t\to\infty}[\hat{z}(t)-z_r(t)]=0$ in the observer interconnection with the reduced system. It follows that system (2) is an asymptotic observer for the reduced system (5).

Conversely, let system (2) be an asymptotic observer for the reduced system (5). Fix $x_{1r}(0)$, $x_{2r}(0)$ and $v(0)$ and let

$$x(0) := T^{-1}\begin{bmatrix}x_{1r}(0)\\ x_{2r}(0)\\ x_3(0)\end{bmatrix}$$

in system (1) where $x_3(0)$ is arbitrary but fixed. Then $x_2 = x_{2r}$ and $\lim_{t\to\infty}x_3(t)=0$ for all choices of $u_r$ and $u$. Define $\delta := x_1 - x_{1r}$ then

$$\dot{\delta} = A_{11}\delta + B_1(u-u_r) + A_{13}x_3, \qquad \delta(0)=0.$$

Since $(A_{11},B_1)$ is controllable, there exists a feedback matrix $F$ such that $A_{11}+B_1F$ is Hurwitz. Consider the auxiliary system

$$\dot{\delta}_F = (A_{11}+B_1F)\delta_F + A_{13}x_3, \qquad \delta_F(0)=0,$$

then $\lim_{t\to\infty}\delta_F(t)=0$ by [3, Corollary 3.22]. Now the choice $u_r := u - F\delta_F$ yields $\delta=\delta_F$ and hence $\lim_{t\to\infty}\delta(t)=\lim_{t\to\infty}[x_1(t)-x_{1r}(t)]=0$. It follows that $\lim_{t\to\infty}[y(t)-y_r(t)]=0$, $\lim_{t\to\infty}[z(t)-z_r(t)]=0$ and $\lim_{t\to\infty}[u(t)-u_r(t)]=0$, i.e. the external behaviors of system (1) and system (5) are asymptotically equal under the above correspondence of trajectories.

According to Proposition 1, the reduced observer (4) is an asymptotic observer for the reduced system (5), and by Lemma 1, $K_{11}$ is Hurwitz: Choose $x(0)=0$ and $u=0$ to obtain $y=0$ and $\lim_{t\to\infty}\hat{z}(t)=0$ for all choices of $v_1(0)$. We can connect this reduced observer also to system (1) instead of to the reduced system (5), and since the external behaviors of these two systems are asymptotically equal, another application of [3, Corollary 3.22] shows that the two resulting observer outputs are asymptotically equal. This implies that the reduced order observer (4) is also an asymptotic observer for system (1) (since $\lim_{t\to\infty}[z(t)-z_r(t)]=0$), and by Proposition 1, so is system (2).

Again, we will simplify the notation in the next section by assuming that system (1) has no stable uncontrollable modes and can recover the general case by replacing the system matrices $(A,B,C,V)$ in the resulting characterization with the reduced system matrices of system (5).

We finish this section by treating the remaining case not covered by Proposition 2, namely the case where system (1) is completely uncontrollable and stable, i.e. where $A$ is Hurwitz and $B=0$.

**Proposition 3.** *Let $A$ be Hurwitz and let $B=0$ in system* (1)*. Then system* (2) *is an asymptotic observer for system* (1) *if and only if $PR(K,M)=0$ and the co-restriction of $K$ to the quotient space $\mathbb{R}^s/\mathfrak{N}(P,K)$ is Hurwitz. Here, $R(K,M)=\left[M\ KM\ \dots\ K^{s-1}M\right]$ is the reachability matrix of the pair $(K,M)$ and $\mathfrak{N}(P,K)\subset\mathbb{R}^s$ is the unobservable subspace of the pair $(P,K)$.*

*Proof.* Let system (2) be an asymptotic observer for system (1) with $A$ Hurwitz and $B=0$. Then $\lim_{t\to\infty}z(t)=0$ and hence $\lim_{t\to\infty}\hat{z}(t)=0$ for all choices of $x(0)$, $v(0)$ and $u$, in particular for $x(0)=0$ (hence $y=0$), and all choices of $v(0)$ and $u$. By Lemma 1 then $PR(K,M)=0$ and the co-restriction of $K$ to the quotient space $\mathbb{R}^s/\mathfrak{N}(P,K)$ is Hurwitz.

Conversely, assume that $PR(K,M)=0$ and that the co-restriction of $K$ to the quotient space $\mathbb{R}^s/\mathfrak{N}(P,K)$ is Hurwitz. Then $P_1R(K_{11},M_1)=0$ and $K_{11}$ is Hurwitz in the reduced observer (4). By Lemma 1, $\lim_{t\to\infty}\hat{z}(t)=0$ for all choices of $v_1(0)$ and $u$ and $y=0$. Since $\lim_{t\to\infty}y(t)=0$ for all choices of $x(0)$ and $u$, an application of [3, Corollary 3.22] yields $\lim_{t\to\infty}\hat{z}(t)=0$ for all choices of $v_1(0)$, $x(0)$ and $u$, and hence the reduced observer (4) is an asymptotic observer for system (1) (since $\lim_{t\to\infty}z(t)=0$). By Proposition 1, so is system (2).

## 3 The main characterization result

We are now in a position to state the main characterization result for asymptotic state observers. We will briefly discuss the error in the previous proof attempt [1, Theorem 9] after we have given the new proof. The proof references the following two technical lemmas proved in [1] that we restate here for convenience but without proof. Consider the linear time-invariant finite-dimensional system in state space form given by

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{6}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$.

**Lemma 2.** *[1, Lemma 3] Let all uncontrollable modes of system* (6) *be unstable. Then, for every $Q \in \mathbb{R}^{q\times n}$ with $Q \neq 0$ there exists an initial condition $x_0$ and an input $u$ such that* $\lim_{t\to\infty} Qx(t) \neq 0$*.*

**Lemma 3.** *[1, Proposition 5] If* $\lim_{t\to\infty} y(t) = 0$ *for all choices of $x_0$ and $u$ in system* (6) *then its transfer function $G(s) = C(sI - A)^{-1}B + D \equiv 0$ and in particular $D = 0$. If, moreover, all uncontrollable modes of system* (6) *are unstable then $C = 0$.*

**Theorem 1.** *Let all uncontrollable modes of system* (1) *be unstable. Then system* (2) *is an observable asymptotic observer for $z$ given $u$ and $y$ if and only if there exists a matrix $U \in \mathbb{R}^{s\times n}$ such that*

$$\begin{aligned} UA - KU - LC &= 0, \\ M - UB &= 0, \\ V - PU - QC &= 0, \end{aligned} \tag{7}$$

*$K$ is Hurwitz and $(P,K)$ is observable.*

*Proof.* Let system (2) be an observable asymptotic observer for $z$ given $u$ and $y$ and define $e := \hat{z} - z$. Then

$$e = Pv + Qy - Vx = Pv - (V - QC)x.$$

Assume, to arrive at a contradiction, that $\mathrm{Im}(V - QC) \not\subset \mathrm{Im}(P)$. Then there exists an invertible $S \in \mathbb{R}^{r\times r}$ such that

$$SP = \begin{bmatrix} P_1 \\ 0 \end{bmatrix} \text{ and } S(V - QC) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

with $V_2 \neq 0$. Now $\lim_{t\to\infty} Se(t) = 0$ implies $\lim_{t\to\infty} V_2 x(t) = 0$ for all initial conditions $x(0)$ and all inputs $u$, a contradiction to Lemma 2. We conclude that $\mathrm{Im}(V - QC) \subset \mathrm{Im}(P)$ and hence there exists a matrix $U \in \mathbb{R}^{s\times n}$ such that $V - QC = PU$. This implies the third equation in (7).

Define $d := v - Ux$ then

$$\begin{aligned}\dot{d} &= \dot{v} - U\dot{x} \\ &= Kv + Ly + Mu - UAx - UBu \\ &= Kv - KUx + KUx + LCx + Mu - UAx - UBu,\end{aligned}$$

and hence the observation error $e = Pv - (V - QC)x = Pv - PUx$ is governed by the error system

$$\begin{aligned}\dot{d} &= Kd - (UA - KU - LC)x + (M - UB)u, \\ e &= Pd.\end{aligned} \tag{8}$$

The first two equations in (7) now follow immediately from an application of Proposition 4 stated below. Apply Lemma 1 to the resulting error system

$$\begin{aligned}\dot{d} &= Kd, \\ e &= Pd\end{aligned}$$

to see that $K$ must be Hurwitz.

Conversely, assume that the system matrices of systems (1) and (2) fulfill Equation (7) with $K$ Hurwitz, then $\lim_{t\to\infty} e(t) = 0$ follows immediately from the form of the error system (8). In its derivation we have only used the third equation in (7).

Before we state and proof the missing Proposition 4 below, let us briefly discuss what is wrong in the proof of [1, Theorem 9]. In that proof, the conclusion after the derivation of the error system (8) uses the argument *[...] follows immediately from [...] the fact that $(P,K)$ is observable (hence $e(t) \to 0$ implies $d(t) \to 0$)*. While this assertion refers only to the error system (8), and is hence actually true *a posteriori*, it is not generally true as an *a priori* assertion about observable systems, which is how it is being used in the logic of the proof given in [1].

By definition, observability of a linear state space system means that zero output *and* input imply zero state, but the property makes no (direct) statement about limits or about the case where the input is nonzero.

One could think that the argument can be saved by the fact that it is only used as an assertion on the totality of *all* solutions of the error system, as in $e(t) \to 0$ for *all* solutions implies $d(t) \to 0$ for *all* solutions. Indeed, Lemma 1 at first seems to support this. Note, however, that Lemma 1 can not be applied to the error system (8), since the two inputs $v = (x,u)$ are not independent signals. In fact, this is the very reason why Theorem 1 is difficult to prove, cf. [1, Remark 8].

Fixing the above error requires the following generalization to [1, Proposition 7].

**Proposition 4.** *Consider the composite system*

$$\begin{aligned}\dot{x} &= Ax + Bu, \\ \dot{d} &= Kd + Rx + Su, \\ e &= Pd,\end{aligned} \tag{9}$$

*and assume that* $\lim_{t\to\infty} e(t) = 0$ *for all choices of* $x(0)$*,* $d(0)$ *and* $u$*. If all uncontrollable modes of* $\dot{x} = Ax + Bu$ *are unstable and* $(P,K)$ *is observable then* $R = 0$ *and* $S = 0$*.*

The proof of this proposition will be given with the help of the following technical lemma.

**Lemma 4.** *Let* $(P,K) \in \mathbb{R}^{r\times s} \times \mathbb{R}^{s\times s}$ *be observable and let* $A \in \mathbb{R}^{n\times n}$ *and* $R \in \mathbb{R}^{s\times n}$*. Then*

$$\begin{bmatrix} 0 & P \end{bmatrix} \begin{bmatrix} A & 0 \\ R & K \end{bmatrix}^i \begin{bmatrix} x \\ 0 \end{bmatrix} = 0 \tag{10}$$

*for all* $x \in \mathbb{R}^n$ *and all* $i \in \mathbb{N}$ *implies* $R = 0$*.*

*Proof.* We have

$$\begin{bmatrix} A & 0 \\ R & K \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ R \end{bmatrix} x$$

and, using (10) with $i = 1$, also $PRx = 0$ for all $x \in \mathbb{R}^n$.

Assume that

$$\begin{bmatrix} A & 0 \\ R & K \end{bmatrix}^i \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} A^i \\ \sum_{l=1}^{i} K^{l-1} R A^{i-l} \end{bmatrix} x \tag{11}$$

for all $x \in \mathbb{R}^n$ and some $i \in \mathbb{N}$ and

$$PK^{l-1}Rx = 0 \tag{12}$$

for all $x \in \mathbb{R}^n$ and all $l = 1, \ldots, i$. Then

$$\begin{aligned} \begin{bmatrix} A & 0 \\ R & K \end{bmatrix}^{i+1} \begin{bmatrix} x \\ 0 \end{bmatrix} &= \begin{bmatrix} A & 0 \\ R & K \end{bmatrix} \begin{bmatrix} A^i \\ \sum_{l=1}^{i} K^{l-1} R A^{i-l} \end{bmatrix} x \\ &= \begin{bmatrix} A^{i+1} \\ RA^i + K \sum_{l=2}^{i+1} K^{(l-1)-1} R A^{(i-(l-1))} \end{bmatrix} x \\ &= \begin{bmatrix} A^{i+1} \\ \sum_{l=1}^{i+1} K^{l-1} R A^{(i+1)-l} \end{bmatrix} x \end{aligned}$$

for all $x \in \mathbb{R}^n$, where we have used hypothesis (11) in the first line. But then (10) implies that

$$0 = P \left( \sum_{l=1}^{i+1} K^{l-1} R A^{(i+1)-l} \right) x = PK^i Rx$$

for all $x \in \mathbb{R}^n$, where we have used hypothesis (12) in the final conclusion.

By induction, it follows that $PK^{i-1}(Rx) = 0$ for all $i \in \mathbb{N}$ and all $x \in \mathbb{R}^n$. By observability of $(P,K)$ this implies $Rx = 0$ for all $x \in \mathbb{R}^n$ and hence $R = 0$.

*Proof (of Proposition 4).* Apply Lemma 3 to system (9) to obtain

$$P(sI-K)^{-1} \left[ -R(sI-A)^{-1}B + S \right] \equiv 0$$

and hence $-R(sI-A)^{-1}B+S\equiv 0$ since $(P,K)$ is observable. Since $S$ is constant and $R(sI-A)^{-1}B$ is strictly proper, it follows that $S=0$ and $R(sI-A)^{-1}B\equiv 0$. If $\dot{x}=Ax+Bu$ was controllable, we would be done at this point, since then $R(sI-A)^{-1}B\equiv 0$ would imply $R=0$. With the help of Lemma 1 and Lemma 4 above we can, however, treat the more general case of this proposition.

Given that all uncontrollable modes of $\dot{x}=Ax+Bu$ are unstable, there exists an invertible $S\in\mathbb{R}^{n\times n}$ such that

$$SAS^{-1}=\begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix} \text{ and } SB=\begin{bmatrix}B_1\\ 0\end{bmatrix},$$

where $A_{11}\in\mathbb{R}^{n_1\times n_1}$, the pair $(A_{11},B_1)$ is controllable and all eigenvalues of $A_{22}\in\mathbb{R}^{n_2\times n_2}$ have non-negative real parts (Kalman decomposition). Define

$$\begin{bmatrix}R_1 & R_2\end{bmatrix}:=RS^{-1} \quad\text{and}\quad \begin{bmatrix}x_1\\ x_2\end{bmatrix}:=Sx,$$

where the block sizes are as for the matrix $SAS^{-1}$ above. Then $R(sI-A)^{-1}B\equiv R_1(sI-A_{11})^{-1}B_1\equiv 0$ and hence $R_1=0$ because $(A_{11},B_1)$ is controllable. It follows that

$$\begin{aligned}\dot{x}_2&=A_{22}x_2, & x_2(0)&=x_{02},\\ \dot{d}&=Kd+R_2x_2, & d(0)&=d_0,\\ e&=Pd.\end{aligned}\tag{13}$$

By Lemma 1, $\lim_{t\to\infty}e(t)=0$ for all choices of $x_{02}$ and $d_0$ in (13) implies that the co-restriction of the linear map

$$\begin{bmatrix}A_{22} & 0\\ R_2 & K\end{bmatrix}:\mathbb{R}^{n_2+s}\to\mathbb{R}^{n_2+s}$$

to the quotient space $\mathbb{R}^{n_2+s}/\mathfrak{N}$ is stable, where

$$\mathfrak{N}:=\mathfrak{N}\left(\begin{bmatrix}0 & P\end{bmatrix},\begin{bmatrix}A_{22} & 0\\ R_2 & K\end{bmatrix}\right)=\bigcap_{i\in\mathbb{N}}\operatorname{Ker}\left(\begin{bmatrix}0 & P\end{bmatrix}\begin{bmatrix}A_{22} & 0\\ R_2 & K\end{bmatrix}^{i-1}\right)$$

denotes the unobservable subspace. A straightforward computation shows that

$$\begin{bmatrix}x_2\\ d\end{bmatrix}\in\mathfrak{N} \qquad\text{implies}\qquad d\in\mathfrak{N}(P,K),$$

i.e. $d=0$ since $(P,K)$ is observable. This shows $\mathfrak{N}\subset\mathbb{R}^{n_2}\times\{0\}$. On the other hand, since all eigenvalues of $A_{22}$ have non-negative real parts, the co-restriction of the above linear map to any quotient space of the form $\mathbb{R}^{n_2+s}/\mathfrak{S}$ with $\mathfrak{S}\subsetneq\mathbb{R}^{n_2}\times\{0\}$ can not be Hurwitz. It follows that $\mathfrak{N}=\mathbb{R}^{n_2}\times\{0\}$. By Lemma 4 above this implies $R=0$.

Note that the use of Proposition 4 eliminates the need for [1, Lemma 6] and with it the use of the theory of pole-zero cancellations, making the final proof of Theorem 1 slightly more elementary.

## 4 Conclusion

The overall picture now is as follows. In the case where $A$ is Hurwitz and $B = 0$ in system (1), the full characterization of asymptotic state observers (2) is given by $PR(K,M) = 0$ and the co-restriction of $K$ to the quotient space $\mathbb{R}^s/\mathfrak{N}(P,K)$ being Hurwitz (Proposition 3). Note that in this case the characterization is independent of the system matrices $(A,B,C,V)$. Otherwise, the characterization is given by Equation (7) and $K$ Hurwitz (Theorem 1), where we may have to replace the system matrices $(A,B,C,V)$ with the reduced system matrices of system (5) if system (1) has stable uncontrollable modes (Proposition 2), and the observer matrices $(K,L,M,P,Q)$ by the observer matrices $(K_{11},L_1,M_1,P_1,Q)$ of the reduced observer (4) if the observer (2) is not observable (Proposition 1).

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