Observers for linear time-varying systems

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Abstract

We give characterizations and necessary and sufficient existence conditions for tracking and asymptotic observers for linear functions of the state of a linear finite-dimensional time-varying state space system. We specialize the results to affine parameter varying systems and bilinear control systems.

Key words: Observers, LTV systems, LPV systems, bilinear systems
1991 MSC: 93C05, 93B07

1 Introduction

In a general sense, causal observation is the problem of finding estimates for the current values of a set of signals given the current and the past values of another set of signals, where both signal sets are interconnected by the action of a dynamical system. Non-causal observation — where future values of the first set of signals may also be used — is sometimes referred to as smoothing in the literature. This paper is only concerned with causal observation. In the work of Luenberger [19,20] a method is described how this can be done in the context of linear finite-dimensional time-invariant state space systems, where the observed signals are the input and the output of the system and the to be estimated signals are linear functions of the state. The main idea is to feed the observed signals into an auxiliary system, the observer, and to use its output as the desired estimate.

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[1] Currently seconded to National ICT Australia Limited, which is funded by the Australian Government's Department of Communications, Information Technology and the Arts and the Australian Research Council through Backing Australia's Ability and the ICT Centre of Excellence Program.

One desired property of such an estimate is that it is asymptotically accurate, in other words, it converges to the actual value of the observed signals when time goes to infinity. An observer that achieves this is usually called an asymptotic observer. A characterisation of all asymptotic (functional) observers, given a controllable linear finite-dimensional time-invariant state space system and a to be observed linear function of its state, has already been given by Fortmann and Williamson [6]. However, their proof is rather incomplete (see [27] for a discussion) and it is only recently that a full proof has been given by Fuhrmann and Helmke [13]. A full characterisation for the existence of such observers has first been given by Schumacher [23], in terms of the existence of conditioned invariant subspaces [2,32] with certain spectral properties, namely outer detectable subspaces [24,31].

Indeed, the outer spectral properties of conditioned invariant subspaces associated to the to be observed linear function of the state determine all possible dynamics of an observer. Willems [30] observed that that spectrum naturally splits into a fixed part associated to a tight subspace (this terminology has been introduced by Fuhrmann and Helmke [12]) and a completely variable part associated to an observability subspace [21,31]. An extensive proof (of the dual result) can be found in Trentelman's thesis [26]. A further analysis of this splitting in a very recent paper by Fuhrmann and the author [14] unearthed the fundamental concept of tracking observer and led to a complete characterisation of all possible observer dynamics, including the dynamic fine structure determined by the invariant factors. This follows from an extension of Rosenbrock's generalized pole placement theorem [22] to the quotient space setting. This generalisation required the careful linking of concepts from Fuhrmann's theory of polynomial models [7] to concepts from geometric control theory, a program started by Fuhrmann and Willems in [15,8] with a first culmination point in the work of Fuhrmann and Helmke [13].

A generalisation of some of these results to the setting of linear behaviors in ARMA form has been given by Valcher and Willems [29], who also rigorously define observers and their desirable properties in terms of sets of observed and to be estimated system variables and their past, current and future values. Again, it is Fuhrmann in a recent preprint [11] who provides a link to the classical results drawing on his concept of a behavior homomorphism [9,10].

The purpose of this paper is to generalise some of the above ideas to the case of linear finite-dimensional time-varying state space systems. A preliminary version of some of the results has been presented at the MTNS in Leuven [28]. The results generalize recent work by Balas et al. [1].

From what has been said above, it becomes evident that Paul Fuhrmann's contributions to observer theory for linear systems are second to none. During the past five years Paul has been a challenging mentor, a vigorous colleague and a friend. I am deeply indebted to him for all the support he provided throughout my short career to date. It is hence a great pleasure to dedicate this paper to him on the occasion of his 70th birthday.

2 Preliminaries

The material in this section can be found in any standard textbook on ordinary differential equations (we used [16] and [5]). We briefly review it in order to introduce our notation. We consider *linear time-varying (LTV) systems* of the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{S}$$

where $A(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n\times n})$, $B(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n\times m})$ and $C(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{p\times n})$ are continuous real matrix functions. It is well known that for each *initial value* $x_0 \in \mathbb{R}^n$ and each continuous *input function* $u(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^m)$ the system (S) has a unique continuously differentiable global solution $x(\cdot; x_0, u) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$ with $x(0; x_0, u) = x_0$, and hence an associated continuous *output function* $y(\cdot; x_0, u) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^p)$.

We are interested in estimates for

$$z(t) = K(t)x(t) \tag{Z}$$

where $K(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{l\times n})$ is a continuous real matrix function and $x(\cdot)$ is a solution of the system (S). We will write $z(\cdot; x_0, u)$ for the (continuous) function $z(\cdot)$ resulting from the solution $x(\cdot; x_0, u)$.

Recall that any solution of the *homogeneous system*

$$\dot{x}(t) = A(t)x(t) \tag{H}$$

associated to (S) can be written as $x(\cdot) = X(\cdot)x(0)$, where $X(\cdot) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^{n\times n})$ is the *principal matrix solution* of (H), i.e. the unique continuously differentiable solution of the matrix initial value problem

$$\dot{X}(t) = A(t)X(t), \quad X(0) = I_n$$

where $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. It is well known that $X(t)$ is invertible for each $t \in \mathbb{R}$ and hence we get the identity

$$\forall_{t\in\mathbb{R}}\; X^{-1}(t)A(t)X(t) - X^{-1}(t)\dot{X}(t) = 0. \tag{1}$$

We will call any continuously differentiable matrix function $T(\cdot) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^{n\times n})$ for which each $T(t)$, $t \in \mathbb{R}$, is invertible a *coordinate transformation* in the *state space* $\mathbb{R}^n$ of system (S). In fact, under the transformation $x'(\cdot) = T^{-1}(\cdot)x(\cdot)$ of solutions $x(\cdot)$ the system (S) is transformed to the system

$$\begin{aligned} \dot{x}'(t) &= A'(t)x'(t) + B'(t)u'(t) \\ y'(t) &= C'(t)x'(t) \end{aligned} \tag{S'}$$

where $A'(\cdot) := T^{-1}(\cdot)A(\cdot)T(\cdot) - T^{-1}(\cdot)\dot{T}(\cdot)$, $B'(\cdot) := T^{-1}(\cdot)B(\cdot)$ and $C'(\cdot) := C(\cdot)T(\cdot)$. It follows from equation (1) that transforming the system (S) with the principal matrix solution $X(\cdot)$ of the associated homogeneous system (H) yields a transformed system (S') with $A'(\cdot) \equiv 0$, i.e. with no internal dynamics.

Recall that a solution $x(\cdot)$ of the homogeneous system (H) is called *attractive for* t_0, where $t_0 \in \mathbb{R}$ is interpreted as *initial time*, if there exists a neighborhood U of $x(t_0) \in \mathbb{R}^n$ such that for each initial value $x_0 \in U$

$$\lim_{t\to\infty} \|x(t; t_0, x_0) - x(t)\| = 0$$

where $x(\cdot; t_0, x_0) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$ denotes the unique continuously differentiable solution of system (H) with $x(t_0; t_0, x_0) = x_0$. Due to linearity, a solution $x(\cdot)$ of the homogeneous system (H) is attractive for time t_0 if and only if the zero solution is attractive for time t_0. Furthermore, since $x(\cdot; t_0, cx_0) = cx(\cdot; t_0, x_0)$ for each $c \in \mathbb{R}$, attractiveness for time t_0 of the zero solution implies

$$\forall_{x_0 \in \mathbb{R}^n} \lim_{t\to\infty} x(t; t_0, x_0) = 0,$$

i.e. *all* solutions of system (H) vanish asymptotically. Conversely, if all solutions of system (H) vanish asymptotically, the zero solution (and hence *any* solution) is attractive for *any* $t_0 \in \mathbb{R}$. It is hence justified to call the matrix function $A(\cdot)$ *attractive* if the zero solution of the associated homogeneous system (H) is attractive for time 0.

Using the principal matrix solution $X(\cdot)$ of system (H) it is immediately clear that $A(\cdot)$ is attractive if and only if

$$\lim_{t\to\infty} \|X(t)\| = 0. \tag{2}$$

Below we state a classical sufficient condition for attractiveness that can be expressed purely in terms of $A(\cdot)$. Daleckiĭ and Kreĭn [5] attribute this result to Bohl [4]. The real matrix function $A : \mathbb{R} \longrightarrow \mathbb{R}^{n\times n}$, $t \mapsto A(t)$ is called *stationary at infinity* if

$$\forall_{\epsilon>0} \exists_{L>0, T>0} \forall_{T \le s < t}\ t - s \le L \Rightarrow \|A(t) - A(s)\| \le \epsilon.$$

Any matrix $C \in \mathbb{R}^{n\times n}$ that occurs as a limit $C = \lim_{k\to\infty} A(t_k)$, where

$(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ is an arbitrary sequence such that $(A(t_k))_{k\in\mathbb{N}} \subset \mathbb{R}^{n\times n}$ converges, is called *ω-limit matrix* of $A(\cdot)$.

Proposition 2.1 *Let $A(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n\times n})$ be bounded and stationary at infinity. If the* upper Bohl exponent

$$\kappa := \sup\{s_{\max}(C) \,|\, C \text{ is } \omega\text{-limit matrix of } A(\cdot)\}$$

is negative then $A(\cdot)$ is attractive. Here $s_{\max}(C)$ denotes the maximal real part of the eigenvalues of C.

Corollary 2.2 *Let $A(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n\times n})$. If $A(\infty) := \lim_{t\to\infty} A(t)$ exists and is stable (all eigenvalues have negative real part) then $A(\cdot)$ is attractive.*

3 Observers for LTV systems

An *observer for* (S) *and* (Z) is an auxiliary system

$$\begin{aligned} \dot{\xi}(t) &= F(t)\xi(t) + G(t)y(t) + H(t)u(t) \\ \zeta(t) &= J(t)\xi(t) \end{aligned} \tag{O}$$

where $F(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{q\times q})$, $G(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{q\times p})$, $H(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{q\times m})$ and $J(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{l\times q})$ are continuous real matrix functions. Again, for every initial value $\xi_0 \in \mathbb{R}^q$ and every pair $u(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^m)$ and $y(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^p)$ of continuous *observer input functions* the system (O) has a unique continuously differentiable global solution $\xi(\cdot; \xi_0, u, y) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^q)$ with $\xi(0; \xi_0, u, y) = \xi_0$, and an associated continuous *observer output function* $\zeta(\cdot; \xi_0, u, y) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^l)$.

We are interested in the *interconnection* of system (S) with the observer (O), i.e. the specific solutions $\xi(\cdot; \xi_0, u, x_0) := \xi(\cdot; \xi_0, u, y(\cdot; x_0, u))$ for initial values $x_0 \in \mathbb{R}^n$ for the system (S), initial values $\xi_0 \in \mathbb{R}^q$ for the observer (O), a common input function u and the second observer input function y being chosen as the resulting output function of the system (S). In this situation we will regard the observer output function $\zeta(\cdot; \xi_0, u, x_0) := \zeta(\cdot; \xi_0, u, y(\cdot; x_0, u))$ as an estimate for the function $z(\cdot; x_0, u)$ in (Z).

There are many different things one could require from such an estimate. E.g., we could require the estimate ζ to converge to z when the time goes to infinity. Such an observer is usually called an *asymptotic (functional) observer*. In the time-invariant case the fundamental property of an observer turns out to be the *tracking property*: for every initial value x_0 of the observed system there exists an initial value ξ_0 of the observer such that for any (continuous) input function u the observer tracks the to be estimated signal, i.e. $\forall_{t\in\mathbb{R}}\ \zeta(t) = z(t)$. This property is fundamental in the sense that it can always be achieved and that for controllable observed time-invariant systems it is already implied by the asymptotic property described before (see [13,14] for a proof). An observer will be called a *tracking observer* if it has the tracking property. A generalisation of the concept of tracking observers to the behavioral setting has been provided by Bisiacco and Valcher in [3]. They call such observers *consistent*.

Definition 3.1 *System* (O) *will be called a* tracking observer for (S) and (Z) *if*

$$\forall_{x_0\in\mathbb{R}^n}\exists_{\xi_0\in\mathbb{R}^q}\forall_{u(\cdot)\in\mathcal{C}(\mathbb{R},\mathbb{R}^m)}\forall_{t\in\mathbb{R}}\ e(t;\xi_0,u,x_0)=0$$

where $e(\cdot;\xi_0,u,x_0) := \zeta(\cdot;\xi_0,u,x_0) - z(\cdot;x_0,u)$ *is the* tracking error.

This means that potentially we could get an *exact* estimate if we knew how to choose ξ_0. Of course, choosing the right ξ_0 is equally hard as exactly estimating $z(0)$, hence this property is mainly of theoretical interest.

Definition 3.2 *System* (O) *will be called an* asymptotic observer for (S) and (Z) *if*

$$\forall_{x_0\in\mathbb{R}^n}\forall_{\xi_0\in\mathbb{R}^q}\forall_{u(\cdot)\in\mathcal{C}(\mathbb{R},\mathbb{R}^m)}\ \lim_{t\to\infty} e(t;\xi_0,u,x_0)=0.$$

In other words, the to be estimated signal is asymptotically identified by the observer.

We observe the following invariance of the tracking property under coordinate transformations in the state space of the observed system (we omit the trivial proof).

Proposition 3.3 *System* (O) *is a tracking observer for* (S) *and* (Z) *if and only if it is a tracking observer for* (S') *and*

$$z'(t) = K'(t)x'(t) \tag{Z'}$$

where $K'(\cdot) := K(\cdot)T(\cdot)$.

We obtain the following necessary condition for a given observer to have the tracking property.

Theorem 3.4 *If system* (O) *is a tracking observer for* (S) *and* (Z) *then there exists a continuously differentiable matrix function* $Z(\cdot)\in\mathcal{C}^1(\mathbb{R},\mathbb{R}^{q\times n})$ *such that*

$$\begin{aligned}\dot{Z} &= FZ - ZA + GC\\ K &= JZ\end{aligned} \tag{T}$$

Furthermore, $Z(\cdot)$ *can be chosen such that tracking is achieved by setting* $\xi_0 := Z(0)x_0$.

Proof. Let system (O) be a tracking observer for (S) and (Z). Observe that $Z(\cdot)$ is a solution of (T) if and only if $Z'(\cdot) := Z(\cdot)T(\cdot)$ is a solution of

$$\begin{aligned}\dot{Z}' &= FZ' - Z'A' + GC' \\ K' &= JZ'\end{aligned}$$

where the other primed matrix functions are those of the transformed system (S') and (Z'). Choosing $T(\cdot) := X(\cdot)$ in Proposition 3.3 we can hence assume w.l.o.g. that $A(\cdot) \equiv 0$.

By the tracking property, for each $x_0 \in \mathbb{R}^n$ there exists a $\xi_0 \in \mathbb{R}^q$ such that for every $u(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^m)$ the tracking error $e(\cdot; \xi_0, u, x_0) \equiv 0$. Denote that ξ_0 by ξ_{0,x_0}. In particular, we get $\zeta(\cdot; \xi_{0,x_0}, 0, x_0) \equiv z(\cdot; x_0, 0)$. Set $u(\cdot) :\equiv 0$, then the solutions for system (S) are just constant functions $x(\cdot; x_0, 0) \equiv x_0$.

Pick a basis $\mathcal{B}$ for the state space $\mathbb{R}^n$ of the observed system (S). Define for each time $t \in \mathbb{R}$ a matrix $Z(t) \in \mathbb{R}^{q\times n}$ by

$$\forall_{x_0 \in \mathcal{B}} \; Z(t)x_0 := \xi(t; \xi_{0,x_0}, 0, x_0)$$

and obtain a continuously differentiable matrix function $Z(\cdot) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^{q\times n})$.

Now pick $x_0 \in \mathcal{B}$. By definition of $Z(\cdot)$ we have

$$\begin{aligned}d_{x_0}(\cdot) &:= \xi(\cdot; \xi_{0,x_0}, 0, x_0) - Z(\cdot)x(\cdot; x_0, 0) \\ &= \xi(\cdot; \xi_{0,x_0}, 0, x_0) - Z(\cdot)x_0 \equiv 0\end{aligned}$$

and hence by the tracking property it follows

$$\begin{aligned}(K(\cdot) - J(\cdot)Z(\cdot))x_0 &= (K(\cdot) - J(\cdot)Z(\cdot))x(\cdot; x_0, 0) \\ &= z(\cdot; x_0, 0) - J(\cdot)Z(\cdot)x(\cdot; x_0, 0) \\ &= \zeta(\cdot; \xi_{0,x_0}, 0, x_0) - J(\cdot)Z(\cdot)x(\cdot; x_0, 0) \\ &= J(\cdot)(\xi(\cdot; \xi_{0,x_0}, 0, x_0) - Z(\cdot)x(\cdot; x_0, 0)) \\ &= J(\cdot)d_{x_0}(\cdot) \equiv 0.\end{aligned}$$

Since $x_0 \in \mathcal{B}$ was arbitrary this implies $K = JZ$.

Pick again $x_0 \in \mathcal{B}$, observe that $0 \equiv d(\cdot) := d_{x_0}(\cdot) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^q)$ and compute

$$\begin{aligned}0 &= \dot{d} \\ &= \dot{\xi} - \dot{Z}x - Z\dot{x} \\ &= F\xi + GCx + Hu - \dot{Z}x - Z(Ax + Bu) \\ &= Fd - (ZA - FZ - GC + \dot{Z})x + (H - ZB)u \\ &= (ZA - FZ - GC + \dot{Z})x.\end{aligned}$$

Here we have dropped the dependencies on time, x_0, ξ_{0,x_0} and $u(\cdot) \equiv 0$ for the sake of readability. We conclude from $x(\cdot; x_0, 0) \equiv x_0$ and from the fact that $x_0 \in \mathcal{B}$ was arbitrary that $ZA - FZ - GC + \dot{Z} = 0$.

The last statement follows from the defintion of $Z(\cdot)$ above since for every $x_0 \in \mathcal{B}$ we have $Z(0)x_0 = \xi_{0,x_0}$. By linearity, for any $x_0 \in \mathbb{R}^n$ the choice $\xi_0 := Z(0)x_0$ will hence achieve tracking. □

The significance of equations (T) is that their solution $Z(\cdot)$ defines an embedding of the system state trajectories into the observer state trajectories of a tracking observer (for $u(\cdot) \equiv 0$). This embedding (if it exists) allows us to write down explicit dynamics for the tracking error $e(\cdot)$ of arbitrary observers (for arbitrary inputs) as follows.

Proposition 3.5 *Let $Z(\cdot)$ be a solution of equations* (T)*. Let $x(\cdot)$ and $\xi(\cdot)$ be arbitrary solutions of* (S) *and* (O)*, respectively. Define $d(\cdot) := \xi(\cdot) - Z(\cdot)x(\cdot)$. Then*

$$\begin{aligned} \dot{d} &= Fd + (H - ZB)u \\ e &= Jd \end{aligned} \tag{E}$$

where $e(\cdot) = \zeta(\cdot) - z(\cdot)$ is the tracking error as above.

The proof follows from a straight forward computation. We can now state a sufficient condition for a given observer to have the tracking property.

Theorem 3.6 *System* (O) *is a tracking observer for* (S) *and* (Z) *if there exists a continuously differentiable matrix function $Z(\cdot) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^{q\times n})$ such that*

$$\begin{aligned} \dot{Z} &= FZ - ZA + GC \\ K &= JZ \\ H &= ZB \end{aligned} \tag{T'}$$

Proof. Let $Z(\cdot)$ be a solution of equations (T'). For a given initial value $x_0 \in \mathbb{R}^n$ of system (S) set the initial value of system (O) to $\xi_0 := Z(0)x_0$. By Proposition 3.5 it follows $e(\cdot) \equiv 0$ since $d(0) = 0$ and $H - ZB = 0$. □

Note that in order to *construct* a tracking observer we only need to solve equations (T) and *define* $H := ZB$.

The gap between the necessary condition and the sufficient condition above can be closed in the case where the *observer* is itself a *completely observable system* on every time interval $[t_0, \infty)$, $t_0 \in \mathbb{R}$. We recall the following definition of complete observability.

Definition 3.7 *Let* $t_0 \in \mathbb{R}$. *A linear time-varying system* (S) *is called* completely observable on the time intervall $[t_0, \infty)$ *if for any pair of initial values* $x_0, x_0' \in \mathbb{R}^n$ *at time* t_0 *and for any pair of input fuctions* $u(\cdot), u'(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^m)$ *with* $\forall_{t\in[t_0,\infty)}\ u(t) = u'(t)$ *the equality*

$$\forall_{t\in[t_0,\infty)}\ y(t; t_0, x_0, u) = y(t; t_0, x_0', u')$$

of the corresponding output functions implies $x_0 = x_0'$.

The interpretation of this condition is that the initial value x_0 at time t_0 is uniquely determined by $u|_{[t_0,\infty)}$ and $y|_{[t_0,\infty)}$ and hence could in principle be determined (observed) from that knowledge. Some of the well known necessary and sufficient conditions for complete observability on finite time intervals (see e.g. [25] and [18] and references therein) can be readily extended to this case. We won't need the exact form of these conditions here, we just note that complete observability depends only on the matrix functions $A(\cdot)$ and $C(\cdot)$, but not on $B(\cdot)$. We can hence also call the pair $(C(\cdot), A(\cdot))$ completely observable. For linear time-invariant systems, i.e. constant matrix pairs (C, A) the condition is equivalent to the usual notion of observability. The relevance of complete observability for the situation at hand is due to the following lemma.

Lemma 3.8 *Let the system* (S) *be completely observable on every time intervall* $[t_0, \infty)$, $t_0 \in \mathbb{R}$. *Then*

$$\forall_{u(\cdot)\in\mathcal{C}(\mathbb{R},\mathbb{R}^m)}\ y(\cdot; 0, u) \equiv 0$$

implies $B(\cdot) \equiv 0$.

Proof. The proof will be done by contradiction. Let $t_0 > 0$ and assume $B(t_0) \neq 0$. Let $X(\cdot)$ be the principal matrix solution of the homogeneous system (H) associated with (S). Then with $M(\cdot) := X^{-1}(\cdot)B(\cdot)$ the matrix $M(t_0) = X^{-1}(t_0)B(t_0) \in \mathbb{R}^{n\times m}$ has a nonzero entry, say $M_{i,j}(t_0)$. By continuity, there exist $m > 0$ and $\epsilon > 0$ such that $\forall_{t\in(t_0-\epsilon,t_0+\epsilon)}\ |M_{i,j}(t)| \geq m$. Here, we can choose ϵ such that $t_0 - \epsilon > 0$. Let $u(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^m)$ be such that all component functions $u_k(\cdot) \equiv 0$ for $k \neq j$, and such that $\forall_{\mathbb{R}\setminus(t_0-\epsilon,t_0+\epsilon)}\ u_j(t) = 0$ but $\int_{t_0-\epsilon}^{t+\epsilon} u_j(s)\mathrm{d}s > 0$. Such a u clearly exists. It follows

$$x_0 := x(t_0 + \varepsilon; 0, u) = X(t_0 + \epsilon) \int_{t_0-\epsilon}^{t_0+\epsilon} M(s)u(s)\mathrm{d}s \neq 0.$$

Consider the second solution $x'(\cdot; 0, 0) \equiv 0$. By uniqueness of solutions we now have a pair of initial conditions x_0 and $x_0' := 0$ at time $t_0 + \epsilon$ and a pair of input functions $u(\cdot)$ and $u'(\cdot) :\equiv 0$ with $\forall_{t\in[t_0+\epsilon,\infty)}\ u(t) = u'(t)$ such that (by assumption) the corresponding output functions $y(\cdot; t_0+\epsilon, x_0, u)$ and $y(\cdot; t_0 + \epsilon, x_0', u')$ both vanish and in particular agree on the time intervall $[t_0 + \epsilon, \infty)$. By complete observability on the time interval $[t_0 + \epsilon, \infty)$ this implies $x_0 = 0$, a contradiction. We hence conclude $B(t_0) = 0$. The argument for $t_0 < 0$ is completely analogous, and $B(0) = 0$ then follows by continuity. □

We can now state the promised necessary and sufficient condition.

Theorem 3.9 *Let the system* (O) *be completely observable on every time interval* $[t_0, \infty)$, $t_0 \in \mathbb{R}$. *Then system* (O) *is a tracking observer for* (S) *and* (Z) *if and only if there exists a continuously differentiable solution* $Z(\cdot) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^{q\times n})$ *of equations* (T').

Proof. Sufficiency follows from Theorem 3.6. For necessity, let system (O) be a tracking observer for (S) and (Z). By Theorem 3.4 there exists a continuously differentiable solution $Z(\cdot) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^{q\times n})$ of equations (T) such that for every initial value $x_0 \in \mathbb{R}^n$ of system (S) the choice $\xi_{0,x_0} := Z(0)x_0$ for the initial value of the observer (O) will lead to

$$\forall_{u(\cdot)\in\mathcal{C}(\mathbb{R},\mathbb{R}^m)}\ e(\cdot;\xi_{0,x_0}, u, x_0) \equiv 0.$$

By Proposition 3.5 the dynamics of the tracking error is governed by system (E) which by assumption is completely observable on every time intervall $[t_{0,\infty})$, $t_0 \in \mathbb{R}$. Note, that (O) and (E) have the matrix functions $F(\cdot)$ and $J(\cdot)$ in common. Apply Lemma 3.8. □

In the case of Theorem 3.6 we have

$$\dot{d}(t) = F(t)d(t)$$

and hence the tracking error $e(\cdot) = J(\cdot)d(\cdot)$ goes to zero whenever $d(\cdot)$ does and $J(\cdot)$ is bounded. We arrive at the following sufficient condition for a tracking observer to be an asymptotic observer.

Corollary 3.10 *A tracking observer with attractive* $F(\cdot)$, *bounded* $J(\cdot)$ *and* $H = ZB$, *where* $Z(\cdot)$ *is a continuosly differentiable solution to equations* (T), *is an asymptotic observer.*

4 Existence conditions

Going back to equations (T') we immediately get the following existence conditions for tracking and asymptotic observers. The conditions make use of the concept of *families of conditioned invariant subspaces* as introduced by Ilchmann [17]. Note that in order to be able to apply all the characterizations of families of conditioned invariant subspaces given by Ilchmann, including the duality results and the characterizations via output injections, we need to restrict our system class to piecewise analytic systems. See Ilchmann's paper for a detailed discussion why this is the case. If we opt for the first equation in (T') as the definition of the family $\operatorname{Ker} Z(\cdot)$ being conditioned invariant (i.e. there exist $F(\cdot)$ and $G(\cdot)$ such that this equation holds), and if we don't require an output injection characterization and a duality result, we can still work within the class of continuous systems. We will call the family $\operatorname{Ker} Z(\cdot)$ *outer detectable* if the matrix function $F(\cdot)$ in the first equation in (T') can be chosen to be attractive.

Theorem 4.1 *There exists a tracking observer for* (S) *and* (Z) *if and only if* $\operatorname{Ker} K(\cdot)$ *contains a conditioned invariant family of subspaces.*

If one of these families is outer detectable then there exists an asymptotic observer for (S) *and* (Z).

In both cases the family of subspaces is given by $\operatorname{Ker} Z(\cdot)$.

Remark 4.2 *We now specialise to systems of the form*

$$\begin{aligned} \dot{x}(t) &= (A_0 + \sum_{i=1}^{r} \rho_i(t) A_i) x(t) + (B_0 + \sum_{i=1}^{r} \rho_i(t) B_i) u(t) \\ y &= (C_0 + \sum_{i=1}^{r} \rho_i(t) C_i) x(t) \end{aligned}$$

where A_i, B_i and C_i are constant matrices and the ρ_i are continuous scalar functions.

If the ρ_i are linearly independent then a particular type of conditioned invariant subspace family is a constant family that is conditioned invariant with respect to all pairs (C_i, A_i), cf. Balas, Bokor and Szabó [1]. That paper also contains algorithms to compute such simultaneously invariant subspaces, which in turn can be used for observer construction by substituting a kernel representation of the subspace into equations (T') *and solving for the observer matrices.*

A further specialisation leads to systems of the form

$$\begin{aligned} \dot{x}(t) &= (A + \sum_{i=1}^{r} u_i(t) B_i) x(t) \\ y &= C x(t) \end{aligned}$$

where A, B_i and C are constant matrices and the controls u_i are interpreted as parameter uncertanties ρ_i. Then apparently the previous result applies. We are hence led to algorithms for an observer construction for bilinear systems. The details of this construction will be reported elsewhere.

5 Acknowledgements

The author wants to thank two anonymous referees for pointing out several weaknesses in exposition of a previous version of this paper.

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