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A converse to the deterministic separation principle

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ABSTRACT

In the classical theory of finite-dimensional linear time-invariant systems in state space form the term *deterministic separation principle* refers to the observation that a stabilizing output feedback controller can be constructed by first constructing an asymptotic state observer that is then coupled to a stabilizing state feedback controller. In this paper we discuss the following *converse* problem: Can *every* stabilizing output feedback controller be realized as interconnection of an asymptotic state observer and a stabilizing state feedback controller? We will provide an affirmative answer to this question (modulo a number of technicalities) in a behavioral setting and with the help of rational representations.

1. Introduction

The classical deterministic separation principle says that, given the plant

$$\dot{x} = Ax + Bu,$$
$$y = Cx,$$

an asymptotic full state observer

$$\dot{\hat{x}} = (A - GC)\hat{x} + Bu + Gy \quad (1)$$

with $A - GC$ Hurwitz, and a stabilizing static full state feedback controller

$$u = F\hat{x} \quad (2)$$

with $A + BF$ Hurwitz, the closed-loop dynamics is given by

$$\begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} A + BF & BF \\ 0 & A - GC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}, \quad (3)$$

where $e = \hat{x} - x$ is the observer error [1]. It follows from the form of the system matrix in (3) that $\lim_{t\to\infty} x(t) = 0$, i.e. the observer-based output feedback controller (1) and (2) is stabilizing. In fact, it is even *internally* or *totally* stabilizing since $x \to 0$ implies both $y \to 0$ and $\hat{x} \to 0$ (since also $e \to 0$), and hence also $u \to 0$.

The above principle is called a *separation* principle because it allows to complete the task of constructing an output feedback controller with desirable properties (namely stability) by *separately* constructing a state observer and a full state feedback controller with that property. Another classical but unrelated separation principle is that of optimal stochastic control, see e.g. [2].

An obvious converse question is whether there are any *other* constructions of (totally) stabilizing output feedback controllers, or whether *any* such controller permits an interpretation as a series connection of a full state observer followed by a (possibly dynamic) full state feedback controller. A partial answer to this question was given by Schumacher using the geometric notion of compensator couples at the beginning of the 1980s [3], but a full answer remained elusive to this date.

In this paper we address the converse question in a behavioral framework using both polynomial and rational representations of linear differential systems. We show that, under mild assumptions on the to be controlled system with variables (x, u, y), any controllable, regular, totally stabilizing controller through the variables (u, y) can be separated into an asymptotic i/o-observer for x from (u, y) with variables $(\hat{x}, u, y)$ and a regular, totally stabilizing controller with variables $(\hat{x}, u)$ in the sense that the controllable part of the observer/$(\hat{x}, u)$-controller interconnection coincides with the given (u, y)-controller.

The paper is organized as follows. Section 2 introduces our notation and collects relevant results from the theory of behaviors including basics on rational representations. In Section 3 we review the required material on stabilization in a behavioral framework. In Section 4, we develop a convenient system representation that

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is adapted to the problem treated in this paper. Section 5 contains the main result, Section 6 discusses the special case of state space systems and Section 7 concludes the paper.

2. Behaviors of linear differential systems

In this paper we will make heavy use of the mathematical machinery of the behavioral approach to linear differential systems. A linear differential system is defined as a triple $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ whose *behavior* $\mathfrak{B} \subset \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is the solution space of a finite set of higher order constant coefficient linear differential equations.

2.1. Polynomial and rational kernel representations

Behaviors of linear differential systems can be represented in terms of a real polynomial matrix $R(s)$ with $\mathtt{w}$ columns as $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, so that

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \right\}. \tag{4}$$

The representation (4) is called a *polynomial kernel representation* of $\mathfrak{B}$, and we often write $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$. If R_1 and R_2 are two full row rank polynomial matrices, then they represent the same behavior $\mathfrak{B}$, i.e. $\mathfrak{B} = \ker R_1(\frac{\mathrm{d}}{\mathrm{d}t}) = \ker R_2(\frac{\mathrm{d}}{\mathrm{d}t})$, if and only if there exists a unimodular polynomial matrix U such that $R_2 = UR_1$. For an extensive treatment of polynomial representations of behaviors we refer to [4].

Behaviors also admit representations in terms of real rational matrices. A detailed exposition on rational representations can be found in [5]. Here we will give a brief review. Recall that any given real rational matrix admits a left coprime factorization into polynomial matrices. A factorization of a real rational matrix R as $R = P^{-1}Q$ with P, Q real polynomial matrices is called a left coprime factorization if $\begin{pmatrix} P & Q \end{pmatrix}$ is left prime (meaning that it has a polynomial right inverse) and $\det(P) \neq 0$. Following [5], if $R = P^{-1}Q$ is such a left coprime factorization then we *define* w to be a solution of $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ if it is a solution of the differential equation $Q(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. In other words, we define

$$\ker R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \ker Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right), \tag{5}$$

which is well-defined since any two left coprime factorizations of R differ by a unimodular polynomial factor. For a given rational matrix R, we call a representation of $\mathfrak{B}$ as $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ a *rational kernel representation* of $\mathfrak{B}$ and write $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$. For additional material on rational representations we refer to [6,7]. In this paper we will often assume that the rational matrices $R(s)$ used in kernel representations have full row rank over the field of real rational functions. This is equivalent to saying that the kernel representation is minimal, see [5,6].

As noted before, two minimal polynomial kernel representations differ by a unimodular polynomial factor. A similar statement does *not* hold for rational representations. We will come back to this in the next subsection.

2.2. Controllability and controllable part

In the behavioral approach an important role is played by the property of *controllability*. The definition of controllability of a behavior $\mathfrak{B}$ is well known, and can be found in [4]. Controllability of a behavior can be tested in terms of its full row rank rational kernel representations as follows: If $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$ where $R(s)$ is a rational matrix, then $\mathfrak{B}$ is controllable if and only if R has no zeros.

A behavior $\mathfrak{B}$ is called *autonomous* if it is a finite dimensional subspace of $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. In terms of its rational kernel representations $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$ this property requires that R has full column rank.

Any behavior $\mathfrak{B}$ admits a direct sum decomposition as $\mathfrak{B} = \mathfrak{B}_{\mathrm{cont}} \oplus \mathfrak{B}_{\mathrm{aut}}$, where $\mathfrak{B}_{\mathrm{cont}}$, called *the controllable part* of $\mathfrak{B}$, is the largest controllable subbehavior of $\mathfrak{B}$, and $\mathfrak{B}_{\mathrm{aut}}$, called *an autonomous part*, is an autonomous subbehavior of $\mathfrak{B}$. The controllable part is uniquely determined by $\mathfrak{B}$. In terms of its rational kernel representations $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, the controllable part of $\mathfrak{B}$ can be found by factorizing $R = Q\bar{R}$ with Q nonsingular rational and $\bar{R}$ a left prime polynomial matrix. For any such factorization we have $\mathfrak{B}_{\mathrm{cont}} = \ker \bar{R}(\frac{\mathrm{d}}{\mathrm{d}t})$, see [5].

It was shown in [6] that if R_1 and R_2 are full row rank rational matrices, then there exists a nonsingular rational matrix Q such that $R_2 = QR_1$ if and only if R_1 and R_2 represent behaviors with the same controllable part, i.e. $(\ker R_1(\frac{\mathrm{d}}{\mathrm{d}t}))_{\mathrm{cont}} = (\ker R_2(\frac{\mathrm{d}}{\mathrm{d}t}))_{\mathrm{cont}}$.

2.3. Elimination of variables

We will now review the basics of elimination of variables. Suppose we have a behavior $\mathfrak{B}$ in which the manifest variable is partitioned into two parts as $w = (v, c)$. Let $R_v(\frac{\mathrm{d}}{\mathrm{d}t})v + R_c(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be a polynomial kernel representation of $\mathfrak{B}$. The space of trajectories that satisfies this equation is called the *full behavior*. The space of tractories c that are compatible with the equation of the full behavior is called *the behavior with v eliminated* and is given by

$$\mathfrak{B}_c := \left\{ c \mid \text{ there exists } v \text{ such that } R_v\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) v + R_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) c = 0 \right\}. \tag{6}$$

The elimination problem is to obtain a kernel representation of (6). Such a kernel representation can be obtained as follows: first find a unimodular polynomial matrix U such that

$$UR_v = \begin{pmatrix} R_{v,1} \\ 0 \end{pmatrix}$$

where $R_{v,1}$ has full row rank. Next, apply the same unimodular matrix to R_c to obtain

$$UR_c = \begin{pmatrix} R_{c,1} \\ R_{c,2} \end{pmatrix}.$$

Since $\ker \begin{pmatrix} R_v & R_c \end{pmatrix}(\frac{\mathrm{d}}{\mathrm{d}t}) = \ker U \begin{pmatrix} R_v & R_c \end{pmatrix}(\frac{\mathrm{d}}{\mathrm{d}t})$, a new, more structured, polynomial kernel representation of $\mathfrak{B}$ is then given by

$$\begin{pmatrix} R_{v,1} & R_{c,1} \\ 0 & R_{c,2} \end{pmatrix} \begin{pmatrix} v \\ c \end{pmatrix} = 0.$$

A kernel representation of the behavior (6) is now given by $R_{c,2}(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ (see [4]).

The above construction to obtain the eliminated behavior (6) is only valid for polynomial kernel representations and uses unimodular premultiplication. Its counterpart for the case that we deal with rational kernel representations and, instead of unimodular premultiplication, we use premultiplication with a nonsingular rational matrix is more subtle and is dealt with in the following lemma.

Lemma 2.1. *Let $R_v(\frac{\mathrm{d}}{\mathrm{d}t})v + R_c(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be a rational kernel representation of $\mathfrak{B}$. Let Q be a nonsingular rational matrix such that*

$$QR_v = \begin{pmatrix} R_{v,1} \\ 0 \end{pmatrix}$$

where $R_{v,1}$ is a full row rank polynomial matrix. Partition

$$QR_c = \begin{pmatrix} R_{c,1} \\ R_{c,2} \end{pmatrix}$$

and assume that $R_{c,1}$ is also a polynomial matrix. Then the controllable parts of $\ker R_{c,2}(\frac{d}{dt})$ and $\mathfrak{B}_c$ are equal.

Proof. Define the rational matrix R by

$$R := \begin{pmatrix} R_{v,1} & R_{c,1} \\ 0 & R_{c,2} \end{pmatrix}. \tag{7}$$

Since $Q(R_v \; R_c) = R$, we have that $\mathfrak{B}_{\text{cont}} = (\ker R(\frac{d}{dt}))_{\text{cont}}$. Let $R_{c,2} = P_2^{-1}Q_2$ be a left coprime factorization. Then $\ker R_{c,2}(\frac{d}{dt}) = \ker Q_2(\frac{d}{dt})$, and also

$$R = \begin{pmatrix} I & 0 \\ 0 & P_2 \end{pmatrix}^{-1} \begin{pmatrix} R_{v,1} & R_{c,1} \\ 0 & Q_2 \end{pmatrix}$$

is a left coprime factorization, so we have

$$\ker R\left(\frac{d}{dt}\right) = \ker \begin{pmatrix} R_{v,1} & R_{c,1} \\ 0 & Q_2 \end{pmatrix}\left(\frac{d}{dt}\right).$$

Since $R_{v,1}$ has full row rank, we have $(\ker R(\frac{d}{dt}))_c = \ker Q_2(\frac{d}{dt}) = \ker R_{c,2}(\frac{d}{dt})$. This yields

$$\begin{aligned}(\mathfrak{B}_c)_{\text{cont}} &= (\mathfrak{B}_{\text{cont}})_c = ((\ker R)_{\text{cont}})_c = ((\ker R)_c)_{\text{cont}} \\ &= \left(\ker R_{c,2}\right)_{\text{cont}},\end{aligned}$$

where we have omitted the differentiation symbol. Here we have used the fact that the operation of taking the controllable part of a behavior, and eliminating a variable from a behavior commute, see e.g. Lemma 2.10.4 in [8]. □

We now study the related question whether the triangular structure in (7) admits a similar triangular structure in a representation of the controllable part. More specifically, assume that the behavior $\mathfrak{B}$ is represented by the rational kernel representation associated with (7), does there exist a representation of its controllable part with compatible triangular structure. This indeed turns out to be the case as is shown in the next lemma.

Lemma 2.2. *Consider the behavior $\mathfrak{B}$ represented by the full row rank rational representation*

$$\begin{pmatrix} R_{v,1}\left(\frac{d}{dt}\right) & R_{c,1}\left(\frac{d}{dt}\right) \\ 0 & R_{c,2}\left(\frac{d}{dt}\right) \end{pmatrix}\begin{pmatrix} v \\ c \end{pmatrix} = 0. \tag{8}$$

Assume that $R_{v,1}$ and $R_{c,1}$ are polynomial, and $R_{v,1}$ has full row rank. Then there exist polynomial matrices $\bar{R}_{v,1}$, $\bar{R}_{c,1}$, $\bar{R}_{c,2}$, and rational matrices $\bar{Q}_{11}$, $\bar{Q}_{12}$ and $\bar{Q}_{22}$, with $\bar{Q}_{11}$ and $\bar{Q}_{22}$ square nonsingular, such that

$$\begin{pmatrix} R_{v,1} & R_{c,1} \\ 0 & R_{c,2} \end{pmatrix} = \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ 0 & \bar{Q}_{22} \end{pmatrix}\begin{pmatrix} \bar{R}_{v,1} & \bar{R}_{c,1} \\ 0 & \bar{R}_{c,2} \end{pmatrix}$$

and such that

$$\begin{pmatrix} \bar{R}_{v,1}\left(\frac{d}{dt}\right) & \bar{R}_{c,1}\left(\frac{d}{dt}\right) \\ 0 & \bar{R}_{c,2}\left(\frac{d}{dt}\right) \end{pmatrix}\begin{pmatrix} v \\ c \end{pmatrix} = 0$$

is a representation of the controllable part $\mathfrak{B}_{\text{cont}}$ of $\mathfrak{B}$.

Proof. First note that since $R_{v,1}$ and $R_{c,1}$ are polynomial matrices, and $R_{v,1}$ has full row rank, $R_{c,2}(\frac{d}{dt})c = 0$ is a kernel representation of $\mathfrak{B}_c$. Factorize

$$\begin{pmatrix} R_{v,1} & R_{c,1} \\ 0 & R_{c,2} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}\begin{pmatrix} \tilde{R}_{v,1} & \tilde{R}_{c,1} \\ \tilde{R}_{v,2} & \tilde{R}_{c,2} \end{pmatrix} \tag{9}$$

where the first factor is nonsingular rational and the second left prime polynomial. The second factor yields a kernel representation of $\mathfrak{B}_{\text{cont}}$. We now eliminate v from $\mathfrak{B}_{\text{cont}}$: by premultiplying the second factor in (9) with a suitable unimodular polynomial matrix we obtain

$$\begin{pmatrix} \bar{R}_{v,1} & \bar{R}_{c,1} \\ 0 & \bar{R}_{c,2} \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} \tilde{R}_{v,1} & \tilde{R}_{c,1} \\ \tilde{R}_{v,2} & \tilde{R}_{c,2} \end{pmatrix}, \tag{10}$$

with $\bar{R}_{v,1}$ full row rank. Then $\bar{R}_{c,2}(\frac{d}{dt})v = 0$ is a representation of $(\mathfrak{B}_{\text{cont}})_c$. By combining (9) and (10) we obtain

$$\begin{pmatrix} R_{v,1} & R_{c,1} \\ 0 & R_{c,2} \end{pmatrix} = \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{pmatrix}\begin{pmatrix} \bar{R}_{v,1} & \bar{R}_{c,1} \\ 0 & \bar{R}_{c,2} \end{pmatrix} \tag{11}$$

for suitable rational matrices $\bar{Q}_{ij}$. From (11) we obtain $\bar{Q}_{21}\bar{R}_{v,1} = 0$. Since $\bar{R}_{v,1}$ has full row rank this yields $\bar{Q}_{21} = 0$. Also we read that $R_{c,2} = \bar{Q}_{22}\bar{R}_{c,2}$. Since $(\mathfrak{B}_c)_{\text{cont}} = (\mathfrak{B}_{\text{cont}})_c$, the controllable part of $\ker R_{c,2}(\frac{d}{dt})$ is equal to $\ker \bar{R}_{c,2}(\frac{d}{dt})$. Since both $R_{c,2}$ and $\bar{R}_{c,2}$ have full row rank, this implies that $\bar{Q}_{22}$ must be square. Then also $\bar{Q}_{11}$ must be square, and both must be nonsingular. □

2.4. Inputs and outputs

Often, the manifest variable w of a behavior $\mathfrak{B}$ is partitioned as $w = \text{col}(u, y)$, and accordingly the rational kernel representation takes the form $R_u(\frac{d}{dt})u + R_y(\frac{d}{dt})y = 0$. This partitioning is called an input–output partition of the behavior if u is input and y is output, see [5]. In terms of the full row rank rational matrices this is equivalent to the property that R_y is a nonsingular rational matrix. In general a behavior admits many input–output partitions. However, the sizes of u and y (denoted by $m(\mathfrak{B})$ and $p(\mathfrak{B})$, respectively) are uniquely determined by the behavior. It is well known that if $R(\frac{d}{dt})w$ is a kernel representation of $\mathfrak{B}$ (either polynomial or rational), then the number of outputs $p(\mathfrak{B})$ of $\mathfrak{B}$ is equal to rank R.

3. Stabilization in a behavioral framework

3.1. Stabilization by full interconnection

Given a behavior $\mathfrak{B}$ (called the plant) with polynomial kernel representation $R(\frac{d}{dt})w = 0$ and a behavior $\mathfrak{C}$ (called a controller) with polynomial kernel representation $C(\frac{d}{dt})w = 0$, the interconnection of $\mathfrak{B}$ and $\mathfrak{C}$ is simply their intersection $\mathfrak{B} \cap \mathfrak{C}$. Clearly, the interconnection has the kernel representation

$$\begin{pmatrix} R\left(\frac{d}{dt}\right) \\ C\left(\frac{d}{dt}\right) \end{pmatrix} w = 0.$$

The interconnection is called *a regular interconnection* if $p(\mathfrak{B} \cap \mathfrak{C}) = p(\mathfrak{B}) + p(\mathfrak{C})$. If both R and C have full row rank, then the interconnection is regular if and only if the matrix

$$\begin{pmatrix} R \\ C \end{pmatrix} \tag{12}$$

has full row rank. The *problem of stabilization by full interconnection* is to find, for $\mathfrak{B}$, a controller $\mathcal{C}$ such that their interconnection is regular, and $\lim_{t\to\infty} w(t) = 0$ for all $w \in \mathfrak{B} \cap \mathcal{C}$. The controller $\mathcal{C}$ is then called a stabilizing controller for $\mathfrak{B}$. If both R and C have full row rank, then $\mathcal{C}$ is a stabilizing controller for $\mathfrak{B}$ if and only if the polynomial matrix (12) is Hurwitz, see [9,10].

Next we will prove that for each stabilizing controller for a given plant, also its controllable part is a stabilizing controller.

Lemma 3.1. *Let $\mathcal{C}$ be a stabilizing controller for $\mathfrak{B}$. Then $\mathcal{C}_{\text{cont}}$ is a stabilizing controller for $\mathfrak{B}$.*

Proof. Let $R(\frac{d}{dt})w = 0$ be a minimal polynomial kernel representation of $\mathfrak{B}$ and let $C(\frac{d}{dt})w = 0$ be a minimal polynomial kernel representation of $\mathcal{C}$. Then the polynomial matrix (12) is Hurwitz. C admits a factorization $C = Q\bar{C}$, with Q nonsingular polynomial and where $\bar{C}(\frac{d}{dt})w = 0$ is a minimal representation of the controllable part $\mathcal{C}_{\text{cont}}$ of $\mathcal{C}$. Then from the fact that

$$\begin{pmatrix} R \\ C \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} R \\ \bar{C} \end{pmatrix}$$

it is easily seen that the matrix

$$\begin{pmatrix} R \\ \bar{C} \end{pmatrix}$$

must be Hurwitz as well. □

The above can be extended to stable *rational* kernel representations for the controller $\mathcal{C}$. We do however adhere in this paper to *polynomial* kernel representations of the plant $\mathfrak{B}$. It is easily seen that if the plant $\mathfrak{B}$ has full row rank polynomial kernel representation $R(\frac{d}{dt})w = 0$ and the controller $\mathcal{C}$ has full row rank rational kernel representation $C(\frac{d}{dt})w = 0$ then $\mathcal{C}$ is a stabilizing controller if and only if the rational matrix (12) is nonsingular and has all its zeros in $\mathbb{C}^-$.

A rational matrix is called a *stable* rational matrix if all its poles lie in $\mathbb{C}^-$. A square nonsingular stable rational matrix is called *miniphase* if its inverse is again stable, equivalently if all its zeros lie in $\mathbb{C}^-$.

If $C(\frac{d}{dt})w = 0$ is a rational kernel representation of the controller $\mathcal{C}$ with C a full row rank *stable* rational matrix, then it is a stabilizing controller if and only if the rational matrix (12) is miniphase, see [5]. For a given plant $\mathfrak{B}$, a stabilizing controller exists if and only if it is *stabilizable*. A definition of stabilizability of behaviors can be found in [4]. If $\mathfrak{B}$ is given by the full row rank rational kernel representation $R(\frac{d}{dt})w = 0$, then $\mathfrak{B}$ is stabilizable if and only if $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}^+$, see [5].

In the following lemma we will give a parametrization of all stabilizing controllers for a given plant $\mathfrak{B}$.

Lemma 3.2. *Let $\mathfrak{B}$ be stabilizable, and let $R(\frac{d}{dt})w = $ be a minimal polynomial kernel representation. Let C_0 be a stable rational matrix such that*

$$\begin{pmatrix} R \\ C_0 \end{pmatrix} \tag{13}$$

is miniphase. Then a controller $\mathcal{C} = \ker C(\frac{d}{dt})$ with C full row rank rational is a stabilizing controller for $\mathfrak{B}$ if and only if there exists a rational matrix F and a square nonsingular rational matrix G with all its zeros in $\mathbb{C}^-$ such that $C = FR + GC_0$.

Proof. Let $C = FR + GC_0$ with F rational and G square nonsingular and all its zeros in $\mathbb{C}^-$. We have

$$\begin{pmatrix} R \\ C \end{pmatrix} = \begin{pmatrix} I & 0 \\ F & G \end{pmatrix} \begin{pmatrix} R \\ C_0 \end{pmatrix},$$

which is clearly square nonsingular and has all its zeros in $\mathbb{C}^-$. Hence $\ker C(\frac{d}{dt})$ is a stabilizing controller. Conversely, let $\ker C(\frac{d}{dt})$ be a stabilizing controller, and C full row rank. Then

$$\begin{pmatrix} R \\ C \end{pmatrix}$$

is square nonsingular and has all its zeros in $\mathbb{C}^-$. Define

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} := \begin{pmatrix} R \\ C \end{pmatrix} \begin{pmatrix} R \\ C_0 \end{pmatrix}^{-1}.$$

This product is clearly square nonsingular and has all its zeros in $\mathbb{C}^-$. Also, it yields $R = Q_{11}R + Q_{12}C_0$, which implies that

$$(I - Q_{11} \quad -Q_{12}) \begin{pmatrix} R \\ C_0 \end{pmatrix} = 0.$$

This clearly implies $Q_{11} = I$ and $Q_{12} = 0$. Now define $F := Q_{21}$, and $G := Q_{22}$. Then G is square nonsingular and has all its zeros in $\mathbb{C}^-$ and, finally, $C = FR + GC_0$. □

3.2. Stabilization by partial interconnection

In general, the plant $\mathfrak{B}$ has two types of variables, the variable w to be controlled, and the variable c, called the interconnection variable, through which the plant can be interconnected with a controller. More specific, if the plant $\mathfrak{B}$ has the polynomial kernel representation $R_w(\frac{d}{dt})w + R_c(\frac{d}{dt})c = 0$, and if a controller $\mathcal{C}$ has the polynomial kernel representation $C(\frac{d}{dt})c = 0$, then the interconnection of $\mathfrak{B}$ and $\mathcal{C}$ through c is defined as the behavior

$$\mathfrak{B} \wedge_c \mathcal{C} := \{(w, c) \mid (w, c) \in \mathfrak{B} \text{ and } c \in \mathcal{C}\}.$$

As before, the interconnection is called regular if $p(\mathfrak{B}) + p(\mathcal{C}) = p(\mathfrak{B} \wedge_c \mathcal{C})$. In this paper we will consider the problem of total stabilization for the plant $\mathfrak{B}$. A controller $\mathcal{C}$ is called a *totally stabilizing controller* for $\mathfrak{B}$ if the interconnection is regular and if $\lim_{t\to\infty}(w(t), c(t)) = 0$ for all $(w, c) \in \mathfrak{B} \wedge_c \mathcal{C}$. If both plant and controller are represented by a minimal polynomial kernel representation, then $\mathcal{C}$ is a totally stabilizing controller for $\mathfrak{B}$ if and only if

$$\begin{pmatrix} R_w & R_c \\ 0 & C \end{pmatrix} \tag{14}$$

is Hurwitz.

Again, the above can be extended to stable *rational* kernel representations for the controller $\mathcal{C}$. We do however again adhere to *polynomial* kernel representations of the plant $\mathfrak{B}$.

If $C(\frac{d}{dt})w = 0$ is a rational kernel representation of the controller $\mathcal{C}$ with C a full row rank stable rational matrix, then it is a totally stabilizing controller if and only if the rational matrix (14) is miniphase. If C is a full row rank rational matrix (not necessarily stable) then it is a totally stabilizing controller if and only if (14) is a square nonsingular rational matrix with all its zeros in $\mathbb{C}^-$.

We will now deal with the question under what conditions there exists a totally stabilizing controller for a given plant $\mathfrak{B}$. These conditions involve stabilizability and detectability. Given the plant behavior $\mathfrak{B}$ with variable (w, c), we say that w is *detectable from* c if $(w, 0) \in \mathfrak{B}$ implies $\lim_{t\to\infty} w(t) = 0$. If $\mathfrak{B}$ is represented by the polynomial kernel representation $R_w(\frac{d}{dt})w + R_c(\frac{d}{dt})c = 0$, then w is detectable from c if and only if $R_w(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^+$, see [4].

The following lemma gives necessary and sufficient conditions on the plant $\mathfrak{B}$ for the existence of a totally stabilizing controller.

Lemma 3.3. *Given a plant $\mathfrak{B}$ with variable (w, c), there exists a totally stabilizing controller $\mathcal{C}$ if and only if $\mathfrak{B}$ is stabilizable and w is detectable from c in $\mathfrak{B}$.*

Proof. Let $R_w(\frac{d}{dt})w + R_c(\frac{d}{dt})c = 0$ be a minimal polynomial kernel representation of $\mathfrak{B}$. By unimodular premultiplication we can obtain an alternative kernel representation in the triangular form

$$\begin{pmatrix} R_{w,1}\left(\frac{d}{dt}\right) & R_{c,1}\left(\frac{d}{dt}\right) \\ 0 & R_{c,2}\left(\frac{d}{dt}\right) \end{pmatrix} \begin{pmatrix} w \\ c \end{pmatrix} = 0, \qquad (15)$$

with $R_{w,1}$ full row rank. Using the assumption that w is detectable from c we find that, in fact, $R_{w,1}(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^+$. As a consequence, $R_{w,1}$ is square and Hurwitz. By stabilizability of $\mathfrak{B}$, $R_{c,2}(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}^+$. Thus, the behavior represented by $R_{c,2}(\frac{d}{dt})c = 0$ itself is stabilizable, so there exists a full row rank polynomial matrix C such that

$$\begin{pmatrix} R_{c,2} \\ C \end{pmatrix}$$

is Hurwitz. Clearly then

$$\begin{pmatrix} R_{w,1} & R_{c,1} \\ 0 & R_{c,2} \\ 0 & C \end{pmatrix} \qquad (16)$$

is Hurwitz, so the controller $\mathcal{C} = \ker C(\frac{d}{dt})$ totally stabilizes $\mathfrak{B}$.

Conversely, assume that there exists C such that

$$\begin{pmatrix} R_w & R_c \\ 0 & C \end{pmatrix}$$

is Hurwitz. Then obviously $(R_w(\lambda)\ R_c(\lambda))$ has full row rank for all $\lambda \in \mathbb{C}^+$, so $\mathfrak{B}$ is stabilizable. Now, assume that $(w, 0) \in \mathfrak{B}$. Then, clearly, also $(w, 0) \in \mathfrak{B} \wedge_c \mathcal{C}$. Since the controller is totally stabilizing we must have $w(t) \to 0$ as $t \to \infty$, so in $\mathfrak{B}$, w is detectable from c. □

4. A convenient system representation

As announced in Section 1, in this paper we will consider linear differential systems $\mathcal{P}$ with system variable (x, u, y), where x is interpreted as the variable to be controlled, and (u, y) as the interconnection variable through which $\mathcal{P}$ can be interconnected with a controller. We will make the following assumptions on $\mathcal{P}$.

(A1) $\mathcal{P}$ is stabilizable,
(A2) x is detectable from (u, y),
(A3) u is input with (x, y) output,
(A4) $\dim(y) \leq \dim(x)$.

We will now briefly discuss the above assumptions. First note that by Lemma 3.3, assumptions (A1) and (A2) are necessary and sufficient for the existence of a totally stabilizing controller for $\mathcal{P}$. In Section 2.3 it was explained that starting from a minimal polynomial kernel representation

$$R_x x + R_u u + R_y y = 0$$

of $\mathcal{P}$, we can always obtain a new, more structured, minimal polynomial kernel representation for $\mathcal{P}$ of the form

$$\begin{pmatrix} R_{x,1} & R_{u,1} & R_{y,1} \\ 0 & R_{u,2} & R_{y,2} \end{pmatrix} \begin{pmatrix} x \\ u \\ y \end{pmatrix} = 0, \qquad (17)$$

in which $R_{x,1}$ has full row rank. We will study how our assumptions are reflected in the polynomial matrices in the representation (17). First note that (A2) holds if and only if $R_{x,1}$ has full column rank for all $\lambda \in \mathbb{C}^+$. Since $R_{x,1}$ has also full row rank we therefore have that assumption (A2) is equivalent with $R_{x,1}$ being Hurwitz. Next, assumption (A3) is equivalent to the condition that the submatrix

$$\begin{pmatrix} R_{x,1} & R_{y,1} \\ 0 & R_{y,2} \end{pmatrix}$$

is square and nonsingular. In particular this implies that $R_{y,2}$ is square and nonsingular. Finally, as a consequence of assumption (A4) we have that the matrix $R_{y,1}$ is tall, meaning that its number of columns does not exceed its number of rows. By applying Lemma 2.2, there exist polynomial matrices $\overline{R}_{x,1}, \overline{R}_{u,1}, \overline{R}_{y,1}, \overline{R}_{u,2}, \overline{R}_{y,2}$ and rational matrices $\overline{Q}_{11}$, $\overline{Q}_{12}$ and $\overline{Q}_{22}$, with $\overline{Q}_{11}$ and $\overline{Q}_{22}$ square nonsingular, such that

$$\begin{pmatrix} R_{x,1} & R_{u,1} & R_{y,1} \\ 0 & R_{u,2} & R_{y,2} \end{pmatrix} = \begin{pmatrix} \overline{Q}_{11} & \overline{Q}_{12} \\ 0 & \overline{Q}_{22} \end{pmatrix} \begin{pmatrix} \overline{R}_{x,1} & \overline{R}_{u,1} & \overline{R}_{y,1} \\ 0 & \overline{R}_{u,2} & \overline{R}_{y,2} \end{pmatrix}$$

and such that

$$\begin{pmatrix} \overline{R}_{x,1} & \overline{R}_{u,1} & \overline{R}_{y,1} \\ 0 & \overline{R}_{u,2} & \overline{R}_{y,2} \end{pmatrix} \begin{pmatrix} v \\ u \\ y \end{pmatrix} = 0 \qquad (18)$$

is a representation of the controllable part $\mathcal{P}_{\text{cont}}$ of $\mathcal{P}$. Obviously, under the assumptions made above $\overline{R}_{x,1}$ and $\overline{R}_{y,2}$ are both nonsingular and $\overline{R}_{y,1}$ is tall.

Remark 4.1. Referring to the discussion at the beginning of Section 3.2, every regular, totally stabilizing controller $\mathcal{C}$ for $\mathcal{P}$ through (u, y) has a polynomial (and hence stable rational) representation $\mathcal{C} = \ker \begin{pmatrix} C_u & C_y \end{pmatrix}$ such that

$$\begin{pmatrix} R_{x,1} & R_{u,1} & R_{y,1} \\ 0 & R_{u,2} & R_{y,2} \\ 0 & C_u & C_y \end{pmatrix}$$

is Hurwitz (and hence miniphase). Since

$$\begin{pmatrix} R_{x,1} & R_{u,1} & R_{y,1} \\ 0 & R_{u,2} & R_{y,2} \\ 0 & C_u & C_y \end{pmatrix} = \begin{pmatrix} \overline{Q}_{11} & \overline{Q}_{12} & 0 \\ 0 & \overline{Q}_{22} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \overline{R}_{x,1} & \overline{R}_{u,1} & \overline{R}_{y,1} \\ 0 & \overline{R}_{u,2} & \overline{R}_{y,2} \\ 0 & C_u & C_y \end{pmatrix}$$

this implies that

$$\begin{pmatrix} \overline{R}_{u,2} & \overline{R}_{u,2} \\ C_u & C_y \end{pmatrix}$$

has full row rank. We will use this fact in the proof of Proposition 5.2.

5. A converse to the separation principle

The following theorem is the main result of this paper. It provides a converse to the classical deterministic separation principle.

Theorem 5.1. *Consider a system with variables (x, u, y) such that Assumptions (A1)–(A4) from Section 4 hold. Then for every controllable, regular, totally stabilizing controller $\mathcal{C}$ through (u, y) there exists an asymptotic i/o-observer $\mathcal{O}$ for x from (u, y) with variables $(\hat{x}, u, y)$ and a regular, totally stabilizing controller $\mathcal{K}$ for the plant system with variables $(\hat{x}, u)$ such that*

$$\left(\mathcal{O} \wedge_{(\hat{x},u)} \mathcal{K}\right)_{(u,y),\text{cont}} = \mathcal{C}. \qquad (19)$$

A few remarks are in order before we prove this theorem. Fig. 1 illustrates the separation of the controller $\mathcal{C}$ into the two blocks $\mathcal{O}$ and $\mathcal{K}$. Assumptions (A1) and (A2) are clearly necessary for totally stabilizing controllers through (u, y) to exist (Lemma 3.3). Assumptions (A3) and (A4) will turn out to be sufficient conditions

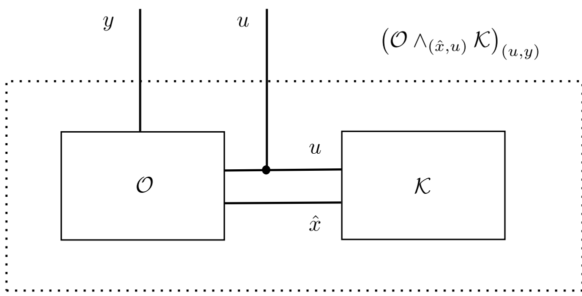

Fig. 1. The separated controller of Theorem 5.1.

for solvability of the matrix equation that is equivalent to (19), cf. Proposition 5.2. Ideally, we would like to drop the assumption of controllability for the controller $\mathcal{C}$ and the corresponding restriction of the theorem statement to the controllable part of the separated controller. This assumption and restriction are, however, necessitated by our use of rational representations. The general theory will likely require more advanced algebraic methods and our inability to generalize the current proof using standard polynomial or rational methods might help to explain why such a result was never obtained in the classical state space literature (cf. [3]). Note, however, that if a controller is totally stabilizing then its controllable part is also totally stabilizing (Lemma 3.1).

In order to prove Theorem 5.1, we first derive a matrix equation formulation of condition (19).

Proposition 5.2. *Consider a system $\mathcal{P}$ with variables (x, u, y) and such that assumptions* (A1) *and* (A2) *from Section* 4 *hold. Represent $\mathcal{P}$ as in* (17). *Consider a controllable, regular, totally stabilizing controller $\mathcal{C} = \ker \begin{pmatrix} C_u & C_y \end{pmatrix}$ through (u, y), where the polynomial representation of $\mathcal{C}$ has been chosen as in Remark* 4.1. *Then there exists an asymptotic i/o-observer $\mathcal{O}$ for x from (u, y) with variables $(\hat{x}, u, y)$ and a regular, totally stabilizing controller $\mathcal{K}$ for $\mathcal{P}$ with variables $(\hat{x}, u)$ such that* (19) *holds if and only if there exists a rational matrix Y and a stable rational matrix T such that*

$$\begin{pmatrix} Y & YT \end{pmatrix} \begin{pmatrix} \overline{R}_{y,1} \\ \overline{R}_{y,2} \end{pmatrix} = -C_y. \tag{20}$$

Proof. The proof of Proposition 5.2 proceeds in four major steps. In a first step we parametrize all observers $\mathcal{O}(S_1, S_2)$ for x from (u, y) using the authors' recent internal model principle for observers [11] and the algebraic generalization thereof [12]. Here, S_1 and S_2 are matrix parameters. In a second step, we use Lemma 3.2 to provide a parametrization $\mathcal{K}(Y_1, Y_2, X)$ of all fully interconnected regular stabilizing controllers for $\mathcal{P}$ where Y_1, Y_2 and X are matrix parameters. In a third step we use Lemma 2.1 to show that the separated controller resulting from the interconnection of $\mathcal{O}(S_1, S_2)$ and $\mathcal{K}(Y_1, Y_2, X)$ has the same controllable part as $\mathcal{C}$ if and only if we choose Y_2 and X appropriately. In a fourth and last step we characterize when the parameter Y_1 can be chosen such that $\mathcal{K}(Y_1, Y_2, X)$ is a controller with variables $(\hat{x}, u)$ as required.

Step 1. Since the system $\mathcal{P}$ is stabilizable, its anti-stabilizable part is equal to its controllable part, cf. Theorem 2.3 in [11]. But then $\mathcal{O} = \ker \begin{pmatrix} \hat{R}_{\hat{x}} & \hat{R}_u & \hat{R}_y \end{pmatrix}$ is a full row rank polynomial representation of an asymptotic i/o-observer for x from (u, y) if and only if $\hat{R}_{\hat{x}}$ is Hurwitz and $\mathcal{P}_{\text{cont}} \subset \mathcal{O}$, cf. Theorem 5.6 in [11] and Corollary 16 in [12]; equivalently,

$$\begin{aligned} \begin{pmatrix} \hat{R}_{\hat{x}} & \hat{R}_u & \hat{R}_y \end{pmatrix} &= \begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} \overline{R}_{x,1} & \overline{R}_{u,1} & \overline{R}_{y,1} \\ 0 & \overline{R}_{u,2} & \overline{R}_{y,2} \end{pmatrix} \\ &= \begin{pmatrix} S_1\overline{R}_{x,1} & S_1\overline{R}_{u,1} + S_2\overline{R}_{u,2} & S_1\overline{R}_{y,1} + S_2\overline{R}_{y,2} \end{pmatrix} \end{aligned}$$

with S_1 polynomial and Hurwitz, and S_2 polynomial. We write $\mathcal{O}(S_1, S_2)$ to indicate the dependence of $\mathcal{O}$ on the matrix parameters S_1 and S_2.

Step 2. By Lemma 3.2, $\mathcal{K} = \ker \begin{pmatrix} K_x & K_u & K_y \end{pmatrix}$ is a fully interconnected, regular, stabilizing controller for $\mathcal{P}$ if and only if

$$\begin{aligned} \begin{pmatrix} K_x & K_u & K_y \end{pmatrix} &= \begin{pmatrix} Z_1 & Z_2 \end{pmatrix} \begin{pmatrix} \overline{Q}_{11} & \overline{Q}_{12} \\ 0 & \overline{Q}_{22} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \overline{R}_{x,1} & \overline{R}_{u,1} & \overline{R}_{y,1} \\ 0 & \overline{R}_{u,2} & \overline{R}_{y,2} \end{pmatrix} + X \begin{pmatrix} 0 & C_u & C_y \end{pmatrix}, \end{aligned}$$

where Z_1 and Z_2 are rational and X is rational and non-singular square with only stable zeros. Define

$$\begin{pmatrix} Y_1 & Y_2 \end{pmatrix} := \begin{pmatrix} Z_1 & Z_2 \end{pmatrix} \begin{pmatrix} \overline{Q}_{11} & \overline{Q}_{12} \\ 0 & \overline{Q}_{22} \end{pmatrix} \begin{pmatrix} \overline{R}_{x,1} & 0 \\ 0 & I \end{pmatrix}$$

and observe that the reparametrization $\begin{pmatrix} Z_1 & Z_2 \end{pmatrix} \leftrightarrow \begin{pmatrix} Y_1 & Y_2 \end{pmatrix}$ is bijective. Hence $\mathcal{K} = \mathcal{K}(Y_1, Y_2, X) = \ker \begin{pmatrix} K_x & K_u & K_y \end{pmatrix}$ is a fully interconnected, regular, stabilizing controller for $\mathcal{P}$ if and only if

$$\begin{aligned} &\begin{pmatrix} K_x & K_u & K_y \end{pmatrix} \\ &= \begin{pmatrix} Y_1 & Y_1\overline{R}_{x,1}^{-1}\overline{R}_{u,1} + Y_2\overline{R}_{u,2} + XC_u & Y_1\overline{R}_{x,1}^{-1}\overline{R}_{y,1} + Y_2\overline{R}_{y,2} + XC_y \end{pmatrix} \end{aligned} \tag{21}$$

with Y_1 and Y_2 rational and X rational and non-singular square with only stable zeros.

Step 3. Compute

$$\begin{aligned} &\begin{pmatrix} I & 0 \\ -Y_1\overline{R}_{x,1}^{-1}S_1^{-1} & I \end{pmatrix} \\ &\times \begin{pmatrix} S_1\overline{R}_{x,1} & S_1\overline{R}_{u,1} + S_2\overline{R}_{u,2} & S_1\overline{R}_{y,1} + S_2\overline{R}_{y,2} \\ Y_1 & Y_1\overline{R}_{x,1}^{-1}\overline{R}_{u,1} + Y_2\overline{R}_{u,2} + XC_u & Y_1\overline{R}_{x,1}^{-1}\overline{R}_{y,1} + Y_2\overline{R}_{y,2} + XC_y \end{pmatrix} \\ &= \begin{pmatrix} S_1\overline{R}_{x,1} & S_1\overline{R}_{u,1} + S_2\overline{R}_{u,2} & S_1\overline{R}_{y,1} + S_2\overline{R}_{y,2} \\ 0 & (Y_2 - Y_1\overline{R}_{x,1}^{-1}S_1^{-1}S_2)\overline{R}_{u,2} + XC_u & (Y_2 - Y_1\overline{R}_{x,1}^{-1}S_1^{-1}S_2)\overline{R}_{y,2} + XC_y \end{pmatrix}. \end{aligned}$$

By Lemma 2.1,

$$\begin{aligned} &(\mathcal{O}(S_1, S_2) \cap \mathcal{K}(Y_1, Y_2, X))_{(u,y),\text{cont}} \\ &= \ker \begin{pmatrix} (Y_2 - Y_1\overline{R}_{x,1}^{-1}S_1^{-1}S_2)\overline{R}_{u,2} + XC_u & (Y_2 - Y_1\overline{R}_{x,1}^{-1}S_1^{-1}S_2)\overline{R}_{y,2} + XC_y \end{pmatrix} \end{aligned}$$

and since

$$\begin{pmatrix} \overline{R}_{u,2} & \overline{R}_{y,2} \\ C_u & C_y \end{pmatrix}$$

has full row rank, $(\mathcal{O}(S_1, S_2) \cap \mathcal{K}(Y_1, Y_2, X))_{(u,y),\text{cont}} = \mathcal{C}$ if and only if $Y_2 = Y_1\overline{R}_{x,1}^{-1}S_1^{-1}S_2$ and X is rational and non-singular square with only stable zeros.

Step 4. It only remains to characterize when $\mathcal{K}(Y_1, Y_1\overline{R}_{x,1}^{-1}S_1^{-1}S_2, X)$ has variables $(\hat{x}, u)$ only, i.e. when $K_y = Y_1\overline{R}_{x,1}^{-1}\overline{R}_{y,1} + Y_2\overline{R}_{y,2} + XC_y = Y_1\overline{R}_{x,1}^{-1}\overline{R}_{y,1} + Y_1\overline{R}_{x,1}^{-1}S_1^{-1}S_2\overline{R}_{y,2} + XC_y = 0$. Setting $Y = X^{-1}Y_1\overline{R}_{x,1}^{-1}$ and $T = S_1^{-1}S_2$ shows that this is equivalent to Eq. (20). □

We can now complete the proof of Theorem 5.1 by showing that, under the conditions of the theorem, Eq. (20) has a solution.

Proof of Theorem 5.1. Assumption (A3) implies that $\overline{R}_{y,2}$ is invertible. By Assumption (A4), the matrix $\overline{R}_{y,1}\overline{R}_{y,2}^{-1}$ is tall and hence there exists a left-invertible rational matrix B such that $T := B - \overline{R}_{y,1}\overline{R}_{y,2}^{-1}$

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is stable rational: let

$$U^{-1}\overline{R}_{y,1}\overline{R}_{y,2}^{-1}V = \begin{pmatrix} \frac{p_1}{q_1} & & & 0 \\ & \ddots & & \vdots \\ & & \frac{p_r}{q_r} & 0 \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

be in Smith–McMillan form then B can be chosen as

$$B := U \begin{pmatrix} -\frac{p_1}{q_1} & & & 0 \\ & \ddots & & \vdots \\ & & -\frac{p_r}{q_r} & 0 \\ 0 & \cdots & 0 & I \\ 0 & \cdots & 0 & 0 \end{pmatrix} V^{-1},$$

making T polynomial and hence stable rational. Let B^L be a left-inverse of B, i.e. $B^L B = I$. Let $Y := -C_y \overline{R}_{y,2}^{-1} B^L$ then

$$\begin{aligned}(Y \quad YT)\begin{pmatrix}\overline{R}_{y,1} \\ \overline{R}_{y,2}\end{pmatrix} &= -C_y \overline{R}_{y,2}^{-1} B^L \overline{R}_{y,1} - C_y \overline{R}_{y,2}^{-1} B^L (B - \overline{R}_{y,1}\overline{R}_{y,2}^{-1})\overline{R}_{y,2} \\ &= -C_y\end{aligned}$$

as required. □

In the above theorem, the only constraints placed on the controller $\mathcal{K}$ are that it is itself regular and totally stabilizing, and that it connects to the plant only through (x, u); $\mathcal{K}$ is written here with variables $(\hat{x}, u)$ since it is intended to be combined with the observer $\mathcal{O}$. In particular, it is not assumed *a priori* that the controller $\mathcal{K}$ acts by *feedback* or that it is even static in the classical sense. The latter would amount to the additional conditions $K_{\hat{x}} = I$ and $K_u = F$ where F is a constant matrix, yielding the controller representation $u = F\hat{x}$. It can be shown that Eq. (20) does not always have a solution under these additional conditions.

6. Example: the state space case

Returning to the state space case that we already briefly discussed in Section 1, assume that our plant $\mathcal{P}$ with system variable (x, u, y) is represented by

$$\dot{x} = Ax + Bu,$$
$$y = Cx,$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. For simplicity, assume that the pair (A, B) is controllable and that the pair (C, A) is detectable. Let $\mathcal{C}$ be a regular totally stabilizing controller through (u, y) and assume that $\mathcal{C}$ is controllable. Let $\mathcal{C}$ have polynomial kernel representation

$$C_u\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)u + C_y\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y = 0.$$

In this section we will identify an asymptotic i/o-observer $\mathcal{O}$ for x through (u, y) and a regular totally stabilizing controller $\mathcal{K}$ through (x, u) such that the controllable part of their interconnection equals $\mathcal{C}$. Clearly, a minimal polynomial kernel representation of $\mathcal{P}$ is given by

$$\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}I - A & B & 0 \\ C & 0 & -I \end{pmatrix}\begin{pmatrix} x \\ u \\ y \end{pmatrix} = 0 \qquad (22)$$

and this reveals that, under the above assumptions on the matrix pairs (A, B) and (C, A), the behavior $\mathcal{P}$ is controllable (so also stabilizable) and that x is detectable from (u, y) in $\mathcal{P}$. Obviously, in $\mathcal{P}$, u is input and (x, y) is output, so the assumptions (A1), (A2) and (A3) of Section 4 hold. Assume that in addition we have $p \le n$ so that also (A4) holds. Since $\mathcal{P}$ is controllable it is equal to its controllable part $\mathcal{P}_{\text{cont}}$. We will now first bring the kernel representation (22) in the required upper diagonal form. Let

$$C(sI - A)^{-1} = L_2^{-1} L_1$$

be a polynomial left coprime factorization, and let N_1 and N_2 be polynomial matrices such that

$$\begin{pmatrix} N_1 & N_2 \\ L_1 & -L_2 \end{pmatrix}$$

is unimodular. Pre-multiplying (22) by this unimodular matrix, we see that an upper diagonal polynomial kernel representation of $\mathcal{P} = \mathcal{P}_{\text{cont}}$ is given by

$$\begin{pmatrix} N_1(\frac{\mathrm{d}}{\mathrm{d}t}I - A) + N_2 C & N_1 B & -N_2 \\ 0 & L_1 B & L_2 \end{pmatrix}\begin{pmatrix} x \\ u \\ y \end{pmatrix} = 0.$$

Denote $R := N_1(sI - A) + N_2 C$ and note that this polynomial matrix is Hurwitz. In order to identify a suitable observer/controller pair we first solve the nonlinear equation (20), that in this case takes the form

$$(Y \quad YT)\begin{pmatrix} -N_2 \\ L_2 \end{pmatrix} = -C_y.$$

Since assumptions (A1)–(A4) hold, this equation indeed has a solution pair (Y, T) with Y rational and T stable rational (cf. the Proof of Theorem 5.1). Next, factorize $T = S_2^{-1} S_1$, with S_1 polynomial and Hurwitz, and S_2 polynomial. This yields an asymptotic i/o-observer $\mathcal{O}$ for x from (u, y) with polynomial kernel representation

$$\begin{pmatrix} S_1 R & (S_1 N_1 + S_2 L_1)B & -S_1 N_2 + S_2 L_2 \end{pmatrix}\begin{pmatrix} \hat{x} \\ u \\ y \end{pmatrix} = 0.$$

It is indeed easily verified that if $(x, u, y) \in \mathcal{P}$ and $(\hat{x}, u, y) \in \mathcal{O}$, then the error $e := \hat{x} - x$ satisfies $S_1 R e = 0$, so $e(t) \to 0$ as $t \to \infty$. Next we identify a regular totally stabilizing controller $\mathcal{K}$ through (x, u). In general, all totally stabilizing controllers for $\mathcal{P}$ are parametrized by (21), with Y_1, Y_2 rational and X nonsingular rational with only stable zeros. Here we take

$$X := I_m, \qquad Y_1 := YR, \qquad Y_2 := YT.$$

This yields the regular totally stabilizing controller $\mathcal{K}$ with rational kernel representation

$$\begin{pmatrix} YR & Y(N_1 + TL_1)B + C_u \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix} = 0.$$

Now interconnect the observer $\mathcal{O}$

$$\begin{pmatrix} S_1 R & (S_1 N_1 + S_2 L_1)B & -S_1 N_2 + S_2 L_2 \end{pmatrix}\begin{pmatrix} \hat{x} \\ u \\ y \end{pmatrix} = 0$$

with $\mathcal{K}$

$$\begin{pmatrix} YR & Y(N_1 + TL_1)B + C_u \end{pmatrix}\begin{pmatrix} \hat{x} \\ u \end{pmatrix} = 0$$

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through the variables $(\hat{x}, u)$. Then the controllable part of the (u, y) behavior of this interconnection is equal to the given controller $\mathcal{C}$ as desired.

7. Conclusions

We have shown that under the mild assumptions (A1)–(A4) from Section 4 the converse of the deterministic separation principle holds true, namely that for a linear differential system with variables (x, u, y) any controllable, regular, totally stabilizing controller through the variables (u, y) can be separated into an asymptotic i/o-observer for x from (u, y) with variables $(\hat{x}, u, y)$ and a regular, totally stabilizing controller with variables $(\hat{x}, u)$ in the sense that the controllable part of the observer/$(\hat{x}, u)$-controller interconnection coincides with the given (u, y)-controller.

Together with the fact that the controllable part of a totally stabilizing controller is also totally stabilizing and combined with the internal model principle for observers and the structure of the interconnection, this result implies that any such controller must contain an internal model of the controllable part of the to be controlled system. This observation should have interesting consequences for robustly stabilizing output feedback controllers.

A characterization of those (u, y)-controllers that can be separated into an asymptotic observer and a *feedback* $(\hat{x}, u)$-controller is the topic of future work as is a generalization to not necessarily controllable (u, y)-controllers and the case $\dim(y) > \dim(x)$.

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