Near-Optimal deterministic filtering on the Rotation Group

Mohammad Zamani, Jochen Trumpf, *Member, IEEE,*
and Robert Mahony, *Senior Member, IEEE*

Abstract—This paper considers the problem of obtaining minimum-energy state estimates for a system defined on the rotation group, $SO(3)$. The signals of the system are modeled as purely deterministic signals. We derive a non-linear observer ("filter") posed directly on $SO(3)$ that respects the geometry of the group and achieves a performance that is close to optimal in the sense of minimizing an integral cost that is measuring the state energy. The performance of the proposed filter is demonstrated in simulations involving large initialization, process and measurement errors where the results are compared against a quaternion implementation of an Extended Kalman Filter (EKF). Our results indicate that the proposed filter achieves better robustness against a range of noise levels and initialization errors.

Index Terms—Deterministic filtering, special orthogonal group, attitude estimation, minimum-energy state estimation.

I. Introduction

O PTIMAL state filtering for noisy state space systems is a core problem in system theory. Optimal filters based on probabilistic modeling frameworks for linear state space systems were developed in the early 1960s (e.g. Kalman [1]). For nonlinear systems, the lack of finite dimensional parameterizations of general stochastic processes has made it in general impossible to find finite dimensional optimal stochastic filters [2]. The most common approach to design practical filters for nonlinear systems is to work with a linearization of the original system, e.g. the Extended Kalman Filter (EKF) [3]. Other methods such as particle filters [4] or the Unscented Kalman Filter (UKF) [5] approximate the infinite dimensional distributions by working with a finite

The authors are with the School of Engineering, Australian National University, Canberra, ACT, Australia. Mohammad.Zamani@anu.edu.au, Jochen.Trumpf@anu.edu.au and Robert.Mahony@anu.edu.au.

sample set. Such sub-optimal methods have enjoyed significant practical success despite the lack of guaranteed global optimality or quantifiable performance bounds except in special cases [6], [7]. Additionally, there are variants of the EKF for special system classes, e.g. for invariant systems [8]. These methods yield symmetry preserving filters and a systematic scheme for tuning them. In the late 1960s, Mortensen [9] proposed the concept of minimum energy filtering based on a deterministic signal setting. For linear systems, the minimum energy optimal filter for the standard quadratic cost function can be realised as an on-line dynamic system that is simply the Kalman filter [10]–[13]. For non-linear systems the deterministic signal framework offers the possibility of measuring the distance to optimality of filters and the hope of developing filters with performance bounds and stability guarantees.

We consider a particular example of a non-linear filtering problem in this paper, that of developing attitude filters on the group of rotation matrices, the special orthogonal group $SO(3)$. A recent survey [14], claimed that the EKF and its variants are still the filters employed in a majority of attitude estimation applications, however, recent work on non-linear observer design for systems with symmetry [15]–[19] has lead to a family of constant gain observers that achieve comparable results to EKF type filters in applications and provide (almost) global stability guarantees. Choukroun et al [20] recently used an embedded representation of the Special Orthogonal Group SO(3) (via the quaternions in $\mathbb{R}^4$) to obtain an 'optimal' filter for the attitude estimation problem, although the filter estimate needs to be re-projected onto the rotation group to obtain feasible estimates. Aguiar et al [21] used the minimum energy filter framework explicitly to show 'optimality' for a system posed on the Spe-

cial Euclidian Group SE(3) using a representation embedded in a vector space, but once again it is required to project the filter state to guarantee feasibility, a process that compromises optimality. Coote et al. [22] considered a simple system on the unit circle S^1 (or $SO(2)$) and used a minimum energy filtering approach to obtain a filter that maintains the state constraint and is near-optimal in the sense that its cost is within a small explicit bound of the optimal cost.

In this paper, we apply the principles of optimal deterministic filtering to an attitude denoising problem (where it is assumed that full measurements of the state are available). Our approach is based on exploiting the geometry of the rotation group and then applying the principles of optimal deterministic filtering. Using an analogous approach to that undertaken by Coote et al. [22] we derive a deterministic filter directly on SO(3) that respects the geometry of the system. The proposed solution is near optimal in the sense that the cost or energy associated with the filter system achieves a cost that is within a certain gap from the optimal cost, and where we provide an explicit bound for the magnitude of the performance gap and argue that it is small in typical conditions. The filter is straightforward to tune, and we believe that it is less sensitive to errors in initial noise covariance estimates than the EKF and its variants. We provide a simulation study that compares the proposed filter against a quaternion implementation of the EKF for large initial state error.

The remainder of the paper is organized as follows. Section II provides a short summary of the notation used throughout the paper. Section III provides the problem formulation. Section IV contains a description of the proposed filter and provides the near-optimality proof. Section V contains the simulation results. A short conclusion and an appendix providing some of the proofs complete the paper.

II. Notation

The rotation group is denoted by SO(3). The associated Lie algebra $\mathfrak{so}(3)$ is the set of skew-symmetric matrices

$$\mathfrak{so}(3) = \{A \in \mathbb{R}^{3\times 3} | A = -A^T\}.$$

The Frobenius norm of a matrix $X \in \mathbb{R}^{3\times 3}$ is given by

$$\|X\| := \sqrt{\operatorname{trace}(X^T X)}.$$

We define a cost $\phi_\Gamma : SO(3) \longrightarrow \mathbb{R}^+$, by

$$\phi_\Gamma(R) := \operatorname{trace}\left[(R-I)^T \Gamma (R-I)\right],$$

where $\Gamma \in \mathbb{R}^{3\times 3}$ is symmetric positive definite. Note that $\phi_\Gamma(R)$ coincides with the squared matrix Frobenius norm of $\Gamma^{\frac{1}{2}}(R-I)$ and hence is non-negative. The projection operator $\mathbb{P} : \mathbb{R}^{3\times 3} \longrightarrow \mathfrak{so}(3)$ is defined by

$$\mathbb{P}(M) := \frac{1}{2}(M - M^T).$$

$I \in \mathbb{R}^{3\times 3}$ is the Identity matrix.

III. Problem Formulation

Consider a system on SO(3)

$$\begin{cases} \dot{R} &= R(A + g\delta), \qquad R(0) = R_0 \\ Y &= R\epsilon \end{cases} \tag{1}$$

where R, Y and ϵ are SO(3) valued state, output and measurement error signals, respectively. The signals A and δ denote the measured angular velocity and the process error, respectively, and take values in $\mathfrak{so}(3)$. The scalar g acts as a scaling for the error signal δ. All signals are considered as deterministic functions of time and we assume sufficient regularity of all signals to ensure existence of unique maximal solutions of the system.

In the minimum energy deterministic filtering approach, we consider hypotheses for the unknown system signals, the initial state R_0, the process error δ, and measurement error ϵ, that are compatible with the system model and the actual observations $A|_{[0,T]}$ and $Y|_{[0,T]}$ (1). Note that this is equivalent to specifying a hypothesis $R_h(t)$ for the state trajectory $R(t)$ that satisfies (1) on $[0,T]$ since such a trajectory is uniquely determined by the unknown signals. The cost of a given hypothesis is measured using the following 'energy' functional (cf. [20]–[22])

$$\begin{aligned} J_T &= \frac{1}{4}\phi_{K_0^{-1}}(R_0) + \int_0^T \left(\frac{1}{2}\|\delta\|^2 + \frac{1}{4}\phi_I(\epsilon)\right) d\tau \\ &= \frac{1}{2}\int_0^T \left(\operatorname{trace}[\delta^T \delta] + \operatorname{trace}[I - \epsilon]\right) d\tau \\ &\quad + \frac{1}{4}\operatorname{trace}\left[(R_0 - I)^T K_0^{-1}(R_0 - I)\right], \end{aligned} \tag{2}$$

where $K_0 \in \mathbb{R}^{3\times3}$ is symmetric positive definite. If the signals associated with a given hypothesis minimize the cost (or energy) function over all possible choices of unknown signals, then the hypothesis is termed optimal. The value $R^*(T)$ of the associated state trajectory is considered as the optimal (minimum-energy) state estimate at time T [9].

Note that at different times, e.g $T_1 < T_2$, this process may yield different optimal state trajectories R_T^*. This is due to the fact that at different times T the trajectory R_T^* is associated with different optimization problems where the same cost (2) is minimized conditioned on different sets of observations [9, p. 389]. At time T_1 the observations $A|_{[0,T_1]}$ and $Y|_{[0,T_1]}$ are used whereas at time T_2 the additional measurements $A|_{[T_1,T_2]}$ and $Y|_{[T_1,T_2]}$ further constrain the entire optimal trajectory.

In general the trajectory of an optimal *filter* will not coincide with any of the optimal hypotheses R_T^*. Indeed, the optimal filter trajectory is the sequence of the final values $R_T^*(T)$ for each T. The goal is now to find a finite dimensional dynamic system depending only on measurement signals that has the sequence of final values $R_t^*(t)$ as its trajectory $\hat{R}(t)$.

IV. MAIN RESULTS

In this section we will discuss our solution to the problem sketched in the previous section. First we define the proposed filter and then we show that this filter is a near-optimal solution to our filtering problem.

Consider the filter

$$\dot{\hat{R}} = \hat{R}\left(A - \mathbb{P}\left(KY^T\hat{R}\right)\right), \qquad (3a)$$

$$\dot{K} = \frac{1}{2}Q - \frac{1}{2}K(Y^T\hat{R}+\hat{R}^TY)K+KA-AK, \quad (3b)$$

where $\hat{R}(0) := I$, $K(0) := K_0$, and $Q \in \mathbb{R}^{3\times3}$ is symmetric positive definite. The signals A and Y are defined by system (1). The filter in Equation (3) consists of two interconnected parts. Equation (3a) evolves on SO(3) and is made from a copy of system (1) plus an innovation term. The innovation term is a weighted distance between the (past) estimated signal and the noisy measured state signal projected on the Lie algebra $\mathfrak{so}(3)$. Note that $Y^T\hat{R}$ is a distance between Y

and $\hat{R}$ on the group SO(3). That is, starting from Y and following this distance we will reach $\hat{R} = Y(Y^T\hat{R})$. The distance can be interpreted in terms of a rotation angle between $\hat{R}$ and Y [18]. The weighting matrix $K \in R^{3\times3}$ is dynamically generated by (3b) and depends on estimates and measurements from the past. Equation (3b) is a time-varying Riccati differential equation. We briefly recall the following facts about the solutions of Riccati equations.

Proposition 1: [23, p. 175] Consider the time-varying matrix Riccati differential equation

$$\dot{K} = Q(t) + KS(t)K + KF(t) + F(t)^TK, \quad (4)$$

with the initial condition $K(0) = K_0$, where $Q(t)$, $S(t)$ and $F(t)$ are continuous functions of time. (i) If $K_0 > 0$, S is symmetric and Q is symmetric positive definite the solution $K(t)$ stays symmetric positive definite for $t > 0$ (as long as it exists). (ii) If Q is symmetric positive semi-definite, S is symmetric negative semi-definite and K_0 is symmetric positive semi-definite then (4) has a solution $K(t)$ for all times $t \geq 0$. This solution is unique, symmetric and positive semi-definite. If K_0 is positive definite then this solution is positive definite.

We now state our main result.

Theorem 1: Consider the system (1) and the cost (2). Given some measurements $Y(t)$ and their associated inputs $A(t)$ for $t \in [0,T]$, assume that unique solutions $\hat{R}(t)$ and $K(t)$ to (3a) and (3b) exist on $[0,T]$. Assuming further that

$$\begin{aligned}
W(T) &:= \int_0^T \left(\frac{1}{4}\operatorname{trace}\left[\frac{1}{4}g^2K^{-2}\left((\hat{R}^TR)^2 - I\right)\right.\right.\\
&\qquad + K^{-1}QK^{-1}\left(I - \hat{R}^TR\right)\\
&\qquad \left.\left.- \hat{R}^TYK\left(\hat{R}^TRK^{-1} - K^{-1}\hat{R}^TR\right)\right]\right)d\tau\\
&\geq 0.
\end{aligned}$$
$$(5)$$

Then the filter (3) yields a near-optimal estimate $\hat{R}(T)$ of the state $R(T)$ in the sense that there exists a hypothesis R_h with $R_h(T) = \hat{R}(T)$ and $J_T \leq J_T^* + W(T)$, where J_T is the functional cost for R_h, J_T^* denotes the optimal value for the cost (2) and $W(T)$ is a bound on the optimality gap.

Proof: Under the conditions listed in the theorem,

$$J_T = \\ \frac{1}{4} \operatorname{trace}\left[(R(T) - \hat{R}(T))K^{-1}(T)(R(T) - \hat{R}(T))^T\right] \\ + \int_0^T \left(\frac{1}{2}\left\|\delta - \frac{1}{4}g(K^{-1}\hat{R}^T R - R^T \hat{R}K^{-1})\right\|^2 \\ + \frac{1}{4}\phi_I(Y^T\hat{R})\right)d\tau + W(T).$$

(6)

Details of this derivation are provided in Appendix A. According to Proposition 1, $K(T)$ is a positive definite matrix and, given the assumption on $W(T)$, it is easily verified that all terms on the right hand side of (6) are non-negative. Thus, the cost function J_T fulfills the inequality

$$J_T \geq \int_0^T \left(\frac{1}{4}\phi_I(Y^T\hat{R})\right)d\tau.$$

(7)

The right hand side of Equation (7) is independent of any specific choice of the variables R_0, $\delta|_{[0,T]}$ and $\epsilon|_{[0,T]}$ and depends only on the measured data $Y(.)$ and the filter signals. Thus, the right hand side of Equation (7) is also a lower bound for the minimum J_T^* of the cost, i.e.

$$J_T^* \geq \frac{1}{4}\int_0^T \phi_I(Y^T\hat{R})d\tau.$$

Consider a hypothesis $R_h : [0, T] \longrightarrow \mathrm{SO}(3)$ for the true trajectory of the system generated by

$$\dot{R}_h = R_h\left(A + \frac{1}{4}g\left(K^{-1}\hat{R}^T R_h - R_h^T \hat{R}K^{-1}\right)\right)$$

(8)

with fixed *final* condition $R_h(T) := \hat{R}(T)$ where $\hat{R}$ and K^{-1} are solutions of the proposed filter (3). It is straightforward to show (by integrating in reverse time) that (8) has a unique initial state $R_h(0)$ that produces the final condition $R_h(T) = \hat{R}(T)$. Define the signal $\epsilon_h : [0, T] \longrightarrow \mathrm{SO}(3)$ by

$$\epsilon_h := R_h^T Y,$$

(9)

and the signal $\delta_h : [0, T] \longrightarrow \mathfrak{so}(3)$ by

$$\delta_h := \frac{1}{4}\left(K^{-1}\hat{R}^T R_h - R_h^T \hat{R}K^{-1}\right).$$

(10)

Equations (8) and (9) show that $R_h(0)$, $\delta_h|_{[0,T]}$ and $\epsilon_h|_{[0,T]}$ together with $A|_{[0,T]}$ and $Y|_{[0,T]}$ satisfy the system equations (1).

Recalling (6) the functional cost J_T of R_h is

$$J_T = \frac{1}{4}\int_0^T \phi_I(Y^T\hat{R})d\tau + W(T) \\ \leq J_T^* + W(T).$$

This completes the proof. ∎

Remark 1: The proposed filter fully specializes to the filter on S^1 by Coote et al [22] once we set $Q = gI$. Particularly, the bound on the optimality gap $W(T)$ also specializes to the one obtained in [22] and in that case is fourth order in the tracking error. In fact, the first two lines of $W(T)$ in (5) complete the square for $\hat{R}^T R$ which is second order in the rotation angle tracking error and hence this part of $W(T)$ is fourth order in the tracking error and in particular small and positive. The third line in (5) is a curvature correction term that in the case of S^1 turns out to be zero. Simulation studies indicate that this term is dominated by the first two terms.

Remark 2: The previous theorem assumes that unique solutions exist for Equation (3) on $[0, T]$. It is not clear *a-priori* that (3b) will always admit a solution on this interval. This could potentially become problematic for (3a) since K appears in this equation as well. We know that according to Proposition 1, for Equation (3b) to have a unique solution the term $(Y^T\hat{R} + \hat{R}^T Y)$ should be positive semi definite. This is the case if and only if the angle of rotation between Y and $\hat{R}$ is less than 90 degrees. It is not trivial to verify this condition in advance as the evolution of $\hat{R}$ also depends on K, however, simulations indicate that this condition holds robustly in practice even for fairly extreme noise conditions.

Remark 3: A key contribution of Theorem 1 lies in providing a bound $W(T)$ given by (5) for evaluating the performance of the filter. This bound is numerically quantifiable and is approximately fourth order in the tracking error. Thus, once the initial transient of the filter is complete, and for moderate modelling error, it is to be expected that the filter will perform qualitatively as well as an optimal filter.

Remark 4: In certain applications, the system considered has no process noise. In such situations the results of this paper can be easily modified to yield simpler filtering formulas.
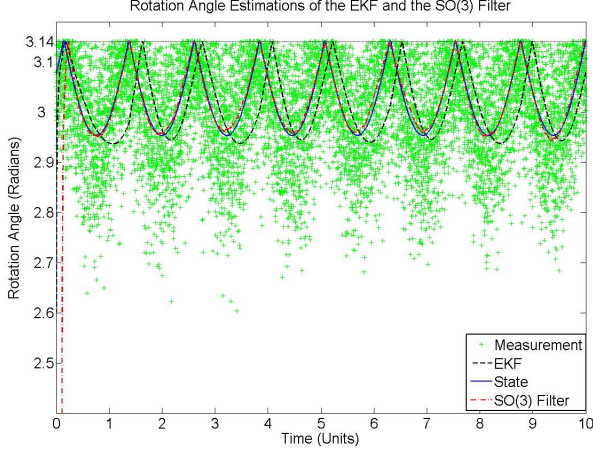
Fig. 1. The rotation angle tracking performance of the proposed filter and the EKF in a typical situation where the EKF performs badly. Note that the figure is zoomed to a neighborhood around an angle of π radians.

V. SIMULATIONS

In this section we present simulation results that demonstrate the performance of the proposed filter. We consider an extreme case for which the initial state error is close to 180 degrees of rotation angle. The EKF is implemented using a quaternion implementation [24]. The numerical solution of the proposed filter is obtained using a RK-MK method, namely the Heun's method [25] modified for a nonlinear feedback system like ours. It is well known that the EKF is prone to singularities when the initial rotation error angle is close to π (cf. [20]) . Figure 1 shows a typical example where the actual state trajectory is initialized and evolves near to π while the two filters are initialized at zero. In this case the majority of the noisy measurements are close to π and due to the inherent singularities in the measurement process , following from the coordinate representation of the EKF, the EKF estimate shows significantly worse performance than the proposed SO(3) filter estimate.

VI. CONCLUSION

In this paper, we proposed a near-optimal deterministic filter that evolves on the rotation group SO(3). The proposed filter shows robustness even to errors of large magnitude and has a quantifiable optimality gap that is asymptotically fourth order in the tracking error, indicating excellent performance and justifying the term "near-

optimal" that we have used to describe the filter. We provide simulations that demonstrate the performance of the proposed filter in a situation with extreme initialization errors.

APPENDIX A
PROOF OF EQUATION (6)

Proof: Consider the function

$$\mathcal{L} = \frac{1}{4} \text{trace} \left[(R - \hat{R}) K^{-1} (R - \hat{R})^T \right]. \quad (11)$$

The time derivative of Equation (11) substituting from Equations (1) and (3) is

$$
\begin{aligned}
\dot{\mathcal{L}} = \text{trace} \Big[\frac{1}{4} \Big(& \left(A + \mathbb{P}(KY^T\hat{R}) \right) \hat{R}^T R \\
& - \hat{R}^T R (A + g\delta) + (A + g\delta) R^T \hat{R} \\
& - R^T \hat{R} \left(A - \mathbb{P}(KY^T\hat{R}) \right) \Big) K^{-1} \\
& - \frac{1}{2} (\hat{R}^T R - I) \dot{K}^{-1} \Big].
\end{aligned}
$$
$$(12)$$

Completing the square for terms containing δ, substituting for $\mathbb{P}(KY^T\hat{R})$, the derivative $\dot{K}^{-1} = -K^{-1}\dot{K}K^{-1}$, the gain dynamics $\dot{K}$ from the Riccati Equation (3b), and finally grouping the remaining terms and extracting $\phi_I(\epsilon)$ and $\phi_I(Y^T\hat{R})$, one obtains

$$
\begin{aligned}
\dot{\mathcal{L}} = & \frac{1}{2} \|\delta\|^2 + \frac{1}{4} \phi_I(\epsilon) - \frac{1}{4} \phi_I(Y^T\hat{R}) \\
& - \frac{1}{2} \left\| \delta - \frac{1}{4} g(K^{-1}\hat{R}^T R - R^T \hat{R} K^{-1}) \right\|^2 \\
& + \text{trace} \Big[\frac{1}{16} K^{-2} g^2 - \frac{1}{16} \left(g K^{-1} \hat{R}^T R \right)^2 \\
& \qquad + \frac{1}{4} K^{-1} Q K^{-1} \hat{R}^T R - \frac{1}{4} Q K^{-2} \\
& \qquad + \frac{1}{4} \hat{R}^T Y K \left(\hat{R}^T R K^{-1} - K^{-1} \hat{R}^T R \right) \Big].
\end{aligned}
$$
$$(13)$$

Integrating $\mathcal{L}$ to obtain $\mathcal{L}(T) - \mathcal{L}(0) = \int_0^T \dot{\mathcal{L}} d\tau$, and replacing $\hat{R}(0) = I$ results in

$$
\begin{aligned}
& \frac{1}{4} \text{trace} \left[(R(T) - \hat{R}(T)) K^{-1}(T) (R(T) - \hat{R}(T))^T \right] \\
& = J_T - \int_0^T \left(\frac{1}{2} \left\| \delta - \frac{1}{4} g(K^{-1}\hat{R}^T R - R^T \hat{R} K^{-1}) \right\|^2 \right. \\
& \qquad\qquad \left. + \frac{1}{4} \phi_I(Y^T\hat{R}) \right) d\tau - W(T).
\end{aligned}
$$
$$(14)$$

Moving J_T to the left hand side yields Equation (6). ∎

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