A non-linear internal model principle for observers

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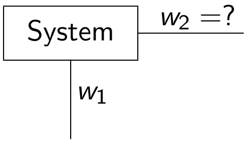
ANU, -

September 2022

Outline

The observation problem

Given a set of *variables* (signals) whose interaction is described by a known *dynamical system* and given *measurements* of some of the variables, can you provide *good estimates* of (other) variables in the system? How?



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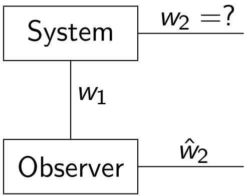

Can you do it with an *observer*?

Observer = system interconnected with the observed system
Estimate = value of variable in the observer
Good observer = observer for which the estimate is satisfactory

Asymptotic state observation

If the observed system is of the form

$$\dot{x} = f(x, u, t)$$
$$y = h(x, u, t)$$

with the variable partition $w_1 = (u, y)$ and $w_2 = x$ then one can try (Luenberger 1964, Kou 1973, Thau 1973) the Ansatz

$$\dot{\hat{x}} = f(\hat{x}, u, t) + \Delta(\hat{x}, y, u, t)$$

and design Δ such that

$$\lim_{t \to \infty} (\hat{x}(t) - x(t)) = 0$$

for all admissible $x(0)$, $\hat{x}(0)$ and u.

The internal model property

Consider an observer of the form

$$\dot{\hat{x}} = f(\hat{x}, u, t) + \Delta(\hat{x}, y, u, t)$$

If $\Delta \equiv 0$ along trajectories of the observed system then this observer is capable of *exactly reproducing* all system trajectories.

Its *behaviour* (= set of trajectories) contains a *full internal model* of the plant behaviour.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu - L(\hat{y} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

If $(A - LC)$ is Hurwitz then $\lim_{t\to\infty}(\hat{x} - x) = 0$ and $\Delta = -L(C\hat{x} - y) \equiv 0$ along plant trajectories where $\hat{x}(0) = x(0)$.

Internal model principles

An *internal model principle for observers* states to what extent the internal model property is *necessary*. Is it true that *every* asymptotic state observer contains a full internal model of the plant?

NO. Consider for example $\dot{x} = Ax$ with A Hurwitz then $\hat{x} = 0$ is an asymptotic observer that only has one single trajectory in common with the plant.

BUT. Every asymptotic LTI observer for an LTI system contains the anti-stabilisable part of the plant behaviour (T./Trentelman/Willems 2014, see also Blumthaler/T. 2014).

For a proof with state space and transfer function methods see *Functional Detectability and Asymptotic Functional Observer Design*, Darouach/Fernando, TAC 2022.

Why internal model principles?

An internal model principle makes a statement about the minimal dynamic complexity necessary for a good observer.

It articulates an obstacle for robust observer design: a good observer must contain sizeable parts of *all* the plant behaviours in the plant set.

An internal model principle can be used to derive *necessary and sufficient* equations for observer design, i.e. a characterization of good observers in terms of equations. In the LTI case this yields the familiar Sylvester equation from Luenberger's work.

Behaviours as sets of trajectories

In this framework a *behaviour* is simply a set of trajectories drawn from a fixed *signal space* $W = W_1 \times W_2$, for example the solution set of an LTI system with variables $w = (w_1, w_2)$, or of a non-linear differential equation on a differentiable manifold with those same variables.

No further structure is assumed a priori, meaning that we can freely choose our favourite notion of solution as long as we obtain all the required properties that are stated later. Trajectories don't even need to be functions of time in our framework.

We must, however, fix the set $\mathcal{B} \subset \mathcal{P}(W_1 \times W_2)$ of behaviours that we will consider in the theory a priori. We will distinguish between sets of signals in general and those sets of signals that are in $\mathcal{B}$ and are called behaviours.

A tale of two relations

We will call the plant behaviour $P \in \mathcal{B}$ and the observer behaviour $O \in \mathcal{B}$.

We assume that we are given two equivalence relations: J on W_1 and T a product relation on W_1 and W_2. We assume $J \subset T$ on W_1.

J (typically $=$) describes the *interconnection* through W_1:

$$P \wedge O = \{((w_1, w_2), (\hat{w}_1, \hat{w}_2)) \in P \times O \,|\, w_1 J \hat{w}_1\}$$

T (for example "asymptotically equal") describes when two signals in W_2 are considered *close* for the purpose of estimation: An observer is good if

$$(P \wedge O)_{(w_2, \hat{w}_2)} \subset T$$

A tale of two relations

Recall that $J \subset T$ on W_1. We can think of T on W_1 as describing a notion of *weak interconnection* and we denote

$$P \wedge_T O = \{((w_1, w_2), (\hat{w}_1, \hat{w}_2)) \in P \times O \mid w_1 \, T \, \hat{w}_1\}$$

We need that for all behaviours $B \in \mathcal{B}$

$$(B \wedge B)_{(w_2,w_2)} \subset T \quad \Longrightarrow \quad (B \wedge_T B)_{(w_2,w_2)} \subset T$$

[This condition is fulfilled if we choose $J = T$ on W_1 and also for the standard choices in the quotient signal space version of the LTI theory ($=$ and "asymptotically equal").]

Saturation of sets of trajectories

Given a set $Q \subset W$ of signals (not necessarily a behaviour in $\mathcal{B}$, we can define its *saturation* under T as

$$\mathrm{Sat}(Q) = \{w' \in W \mid \exists_{w \in Q} \; w' T w\}$$

Obviously $Q \subset \mathrm{Sat}(Q)$ and a set Q with $\mathrm{Sat}(Q) \subset Q$ is called *saturated*.

[This construction replaces localisation at T in Oberst's algebraic analysis framework since at our level of generality there is no longer a useful notion of quotient signal space.]

Nonintrusive observers

We do not want an observer to change the plant behaviour when it is connected to the plant. Formally, we call an observer *nonintrusive* if

$$P \subset (P \wedge O)_w$$

An approximate internal model principle

Under the above conditions, a non-intrusive observer is good if and only if $P \wedge O$ has a T-autonomous hidden behaviour, i.e.

$$\forall_{(w_1,w_2,\hat{w}_2),(w_1',w_2',\hat{w}_2')\in P\wedge O}\; w_1 J w_1' \implies \hat{w}_2 T \hat{w}_2'$$

and

$$P \subset \mathrm{Sat}((P \wedge O)_{\hat{w}})$$

[In the LTI case this can be formulated as $\hat{w}_2$ is detectable from $\hat{w}_1$ in O and $P_T \subset O_T$.]

The radical set of a behaviour

Given a behaviour $P \in \mathcal{B}$, its *radical set* is given by

$$P^T = \{Q \in \mathcal{B} \,|\, \mathrm{Sat}(Q) = \mathrm{Sat}(P)\}$$

Since $P \in P^T$ the radical sets of behaviours are all non-empty and form a poset

$$\mathbb{B} = \{P^T \mid P \in \mathcal{B}\}$$

with the partial order given by

$$P_1^T \leq P_2^T \;\Leftrightarrow\; \forall_{Q_2 \in P_2^T} \exists_{Q_1 \in P_1^T} Q_1 \subset Q_2$$

Local poset sections

The set of saturated behaviours

$$\mathrm{Sat}(\mathcal{B}) = \{\mathrm{Sat}(P) \mid P \in \mathcal{B}\}$$

is partially ordered by set inclusion.

We say that the poset of radical sets $\mathbb{B}$ *allows local poset sections* if the map

$$\mathrm{Sat}(\mathcal{B}) \longrightarrow \mathbb{B}, \quad \mathrm{Sat}(P) \mapsto P^T$$

is an isomorphism of posets.

[This holds in the LTI case and for kinematic systems on differentiable manifolds.]

An internal model principle

A poset is called *well-founded* if every descending chain of elements has a lower bound. In this case the poset contains minimal elements.

Theorem: If $\mathbb{B}$ allows local poset sections and all radical sets are well-founded then every nonintrusive good observer contains a minimal element of the radical set of the plant.

[In the LTI case the minimal element of the radical set is unique and equals the anti-stabilisable part of the plant.]

Kinematic systems on differentiable manifolds

We fix a differentiable manifold M that will serve as the state space of the plant. The plant behaviour P with variables (x, v, y) is the solution set of the system of equations

$$\dot{x} = F_P(x, v)$$
$$y = h(x)$$

where $x \in M$, $v \in V$ with V a vector space, and $y \in Y$ with Y a topological space.

We require F_P to be continuous, *linear* in v (*kinematic system*), and to allow unique C^1-solutions for all admissible inputs (must include all continuous inputs) as long as the resulting solution avoids a fixed *exceptional set* $E_0 \subset M$. h must be continuous.

Candidate observer behaviours

For any differentiable manifold L with a differentiable map $\pi\colon L \to M$ with closed image we choose a set of continuous vector fields F_O such that the associated observer equations

$$\dot{z} = F_O(z, v, y)$$
$$\hat{x} = \pi(z)$$

have unique C^1-solutions for any (v, y) generated by the plant with solutions depending differentiably on initial conditions.

The *observer behaviour* O associated with F_O is then the projected behaviour with variables $(\hat{x}, v, y)$ on the state manifold M. In this setup the plant behaviour P itself is also a candidate observer.

Observer errors

In order to compare the observer estimate $\hat{x} \in M$ and the true system state $x \in M$ we need an error map

$$\mathcal{E}\colon M \times M \to M$$

with the *2-out-of-3 property*: The equation $e = \mathcal{E}(\hat{x}, x)$ is uniquely solvable given any two of the variables and the solution depends differentiably on all the data.

Classical examples of such maps are $e = \hat{x} - x$ in $\mathbb{R}^n$ or $E = \hat{X}^{-1}X$ on a Lie group.

We choose a *zero error state* $e_0 \in M$ with $e_0 \notin E_0$, J and T on v as equality, and T on x (resp. $\hat{x}$) as

$$\hat{x}\, T x \;\Leftrightarrow\; \lim_{t\to\infty} \mathcal{E}(\hat{x}(t), x(t)) = e_0$$

A non-linear internal model principle

It is comparatively easy to show that if $\pi \colon L \to M$ is not surjective then $O \notin P^T$. This implies that a good observer can't be constructed on a manifold L of lower dimension than M.

Theorem: If the poset $\mathcal{B}$ of behaviours is restricted to manifolds L for which $\pi \colon L \to M$ is a submersion with compact fibers then $P^T = \{P\}$.

Corollary: Given a kinematic system on M, any nonintrusive good observer designed on a manifold L for which $\pi \colon L \to M$ is a submersion with compact fibers contains a full internal model of the plant.

[This is the case for a symmetry Lie group acting properly and transitively on M as in the observer theory the first author presented at MTNS 2018.]

What goes wrong with non-compact fibers?

Consider the plant

$$\dot{x} = v$$
$$y = x$$

and the observer

$$\dot{z}_1 = v - (z_1 - y) + e^{-z_2}$$
$$\dot{z}_2 = |z_2| + 1$$
$$\hat{x} = z_1$$

Conjecture: Any nonintrusive good observer for a kinematic system contains a full internal model of the plant "shifted by a fixed small trajectory" (to be made precise).

Conclusion

- introduced a set-theoretic framework for observer theory
- formulated an extremely general approximate internal model principle for observers
- used poset theory to state a set-theoretic internal model principle for observers
- the known LTI results are a special case
- applied the theory to kinematic systems on manifolds to obtain a non-linear internal model principle applicable to observers on symmetry Lie groups
- TODO: non-compact fibers and translation to algebraic characterisation of observers

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Thank you.