A manifold structure on the set of functional observers

Jochen Trumpf

University of Würzburg

Math. Institute

Germany

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joint work with U. Helmke

Motivating problem

Definition. Let $(C, A) \in \mathbb{R}^{p\times n} \times \mathbb{R}^{n\times n}$. A linear subspace $\mathcal{V} \subset \mathbb{R}^n$ is called (C, A)-*invariant* if there exists an output injection matrix J such that

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cf. related work on stable subspaces by

- L. Rodman (various articles) or
- F. Velasco (LAA 301, pp. 15–49, 1999)

Motivating problem

Let $P \in \mathbb{R}^{n \times n}$ be the orthogonal projector on $\mathcal{V}$. Then

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Let (P_0, J_0) be such that $f(P_0, J_0) = 0$. Consider

$$\frac{\partial f}{\partial P}|_{(P_0,J_0)}(\dot{P}) = -\dot{P}A_0P_0 + (I_n - P_0)A_0\dot{P}, \quad A_0 := A - J_0C$$

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in the basis where

$$P_0 = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}, A_0 = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} \text{ and } \dot{P} = [P_0, \Omega] = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}$$

(Ω is skew-symmetric, here.)

Motivating problem

We get

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Result. Let $f(P_0, J_0) = 0$ and

$$\sigma(A_0|_{\operatorname{Im} P_0}) \cap \sigma(A_0|_{\mathbb{R}^n/\operatorname{Im} P_0}) = \emptyset$$

Then locally around J_0 there exists a Lipschitz continuous function $J \mapsto P(J)$ such that

$$f(J, P(J)) = 0$$

Tracking observers

Consider the linear, time-invariant, finite-dimensional control system in state space form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \text{(sys)}$$

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Definition. A *tracking observer* for Vx is a dynamical system

$$\dot{v} = Kv + Ly + Mu \qquad \text{(obs)}$$

which is driven by u and by y and has the *tracking property*:

$$v(0) := Vx(0) \;\Rightarrow\; v(t) = Vx(t) \quad \text{for all } t \in \mathbb{R}$$

where $x(0)$ and $u(.)$ are arbitrary.

Tracking observers

Theorem. (Luenberger, 1964) System (obs) is a tracking observer for Vx if and only if

$$\begin{aligned} VA - KV &= LC \\ M &= VB \end{aligned} \qquad \text{(syl)}$$

In this case the *tracking error* $e(t) = v(t) - Vx(t)$ is governed by the differential equation $\dot{e} = Ke$.

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Theorem. (Willems et al., ≈ 1980) Let V be of full row rank. For every tracking observer for Vx there exists a friend J of $\mathrm{Ker}\, V$ such that $(A - JC)|_{\mathbb{R}^n / \mathrm{Ker}\, V}$ is similar to K. Conversely, for every friend J of $\mathrm{Ker}\, V$ there exists a unique tracking observer for Vx such that K is similar to $(A - JC)|_{\mathbb{R}^n / \mathrm{Ker}\, V}$. Especially, there exists a tracking observer for Vx if and only if $\mathrm{Ker}\, V$ is (C, A)-invariant.

The manifold of tracking observers

Theorem. (T., 2002) Let (C, A) be observable and let k and p be the numbers of rows of V and C, respectively. Then the set

$$\mathrm{Obs}_{k,k} := \{(K, L, M, V) \,|\, VA - KV = LC, M = VB, \mathrm{rk}\, V = k\}$$

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Proof. The value $(0, 0)$ is a regular value of the map

$$f : \ (K, L, M, V) \mapsto (VA - KV - LC, M - VB)$$

The requirement $\mathrm{rk}\, V = k$ yields an open subset.

The manifold of tracking observers

Theorem. (T., 2002) Consider the similarity action

$$\sigma : \mathrm{GL}(k) \times \mathrm{Obs}_{k,k} \longrightarrow \mathrm{Obs}_{k,k} \ ,$$
$$(S, (K, L, M, V)) \mapsto (SKS^{-1}, SL, SM, SV)$$

The σ-orbit space $\mathrm{Obs}^{\sigma}_{k,k}$ of similarity classes

$$[K, L, M, V]_\sigma = \{(SKS^{-1}, SL, SM, SV) \,|\, S \in \mathrm{GL}(k)\} \ .$$

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Proof. The equations $VA - KV = LC$ and $M = VB$ are invariant under σ. The similarity action is free and has a closed graph mapping. Furthermore, $\dim \mathrm{GL}(k) = k^2$.

Conditioned invariants and friends

Theorem. (Helmke/T., 2002) Let (C, A) be observable, let $p \times n$ be the format of C and let $0 \leq k < n$. Then the set

$$\mathrm{Inv}_k = \{(\mathcal{V}, J) \,|\, (A - JC)\mathcal{V} \subset \mathcal{V}, \operatorname{codim} \mathcal{V} = k\}$$

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Proof. http://statistik.mathematik.uni-wuerzburg.de/~jochen

Application: OAF-compensators

One way of stabilizing system (sys) is to dynamically feed back the state v of an appropriately designed tracking observer (obs) via

$$u = Fv + r$$

Here the observer matrix K as well as $(A + BF)$ have to be stable. r denotes an external reference signal.

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$\Longrightarrow$ Minimize the L^2-sensitivity of the closed loop transfer function from r to y over the previously defined observer manifold to get the OAF-compensator best suited to fixed point arithmetics as used in hardware signal processors.

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Thank you.