A Bruhat type decomposition of the set of conditioned invariant subspaces

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output injection

Given a pair of matrices $(C, A) \in \mathbb{R}^{p\times n} \times \mathbb{R}^{n\times n}$ the group action

$$(T, J, S), (C, A) \mapsto (SCT^{-1}, T(A - JC)T^{-1})$$

where $J \in \mathbb{R}^{n\times p}$ is arbitrary and $T \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{p\times p}$ are invertible is called the *output injection equivalence* action.

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Its corresponding normal form is the so-called *Brunovsky normal form*. In the case where (C, A) is observable it looks like this.

Brunovsky form

$$C = \left(\begin{array}{c|c|c} 0\ldots 0\,1 & & \\ \hline & \ddots & \\ \hline & & 0\ldots 0\,1 \end{array}\right)_{\displaystyle \underbrace{\qquad\quad}_{\mu_1} \;\ldots\; \underbrace{\qquad\quad}_{\mu_p}} \quad \textbf{and} \quad A = \left(\begin{array}{c|c|c} \begin{matrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix} & & \\ \hline & \ddots & \\ \hline & & \begin{matrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix} \end{array}\right)_{\displaystyle \underbrace{\qquad\qquad}_{\mu_1} \;\ldots\; \underbrace{\qquad\qquad}_{\mu_p}}$$

where $\mu_1 \geq \ldots \geq \mu_p \geq 0$ are the *observability indices* of ther pair (C, A) which form a complete set of invariants.

restrictions

A *conditioned invariant* or (C, A)-*invariant* subspace is a subspace $\mathcal{V} \subset \mathbb{R}^n$ for which there exists an output injection J such that

$$(A - JC)\mathcal{V} \subset \mathcal{V}$$

Note that this concept is invariant under output injection equivalence.

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There is a natural notion of *restriction* of the pair (C, A) to $\mathcal{V}$ as follows.

restrictions

$$\begin{tikzcd}
\mathbb{R}^n/\mathcal{V} \arrow[r, "\tilde{A}"] & \mathbb{R}^n/\mathcal{V} \arrow[r, "\tilde{C}"] & \mathbb{R}^p/C(\mathcal{V}) \\
\mathbb{R}^n \arrow[u, two heads] \arrow[r, "A - JC"] & \mathbb{R}^n \arrow[u, two heads] \arrow[r, "C"] & \mathbb{R}^p \arrow[u, two heads] \\
\mathcal{V} \arrow[u, hook] \arrow[r, "\bar{A}"] & \mathcal{V} \arrow[u, hook] \arrow[r, "\bar{C}"] & C(\mathcal{V}) \arrow[u, hook]
\end{tikzcd}$$

restriction indices

It can be shown that all matrix representations $(\bar{C}, \bar{A})$ of restrictions are output injection equivalent and that all pairs that are output injection equivalent to a given matrix representation of a restriction are in fact such.

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Hence there is a matrix representation of a restriction in Brunovsky normal form. Its indices $(\lambda_1, \ldots, \lambda_p)$ are called the *restriction indices* of (C, A) with respect to $\mathcal{V}$. It is $\lambda_i \leq \mu_i$ for $i = 1, \ldots, p$.

image representations

Theorem [FPP]:

The codimension q (C,A)-invariant subspaces are precisely the images of full rank matrices $Z \in \mathbb{R}^{n\times(n-q)}$ which fullfill

(1) $\mathsf{A}Z = Z\bar{A} + \mathsf{A}Z\bar{C}^{\top}\bar{C}$

(2) $\mathsf{C}Z = \mathsf{C}Z\bar{C}^{\top}\bar{C}$

(3) $\mathsf{C}Z\bar{C}^{\top}$ **has full rank**

where $(\bar{C},\bar{A})$ is the matrix representation of a restriction in Brunovsky form.

image representations

The set of all these Zs is called $M(\lambda, \mu)$, we denote $\Gamma(\lambda) = M(\lambda, \lambda)$ which is a group. In fact $\operatorname{Im} Z_1 = \operatorname{Im} Z_2$ for $Z_1, Z_2 \in M(\lambda, \mu)$ if and only if $Z_2 = Z_1 S^{-1}$ for $S \in \Gamma(\lambda)$.

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Hence the *Brunovsky stratum* of all codimension q conditioned invariant subspaces with fixed restriction indices λ is a smooth manifold diffeomorphic to $M(\lambda, \mu)/\Gamma(\lambda)$.

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Equations (1) and (2) can be solved explicitely.

image representations

$$Z = (Z_{ij}),\ Z_{ij} = \begin{cases} \begin{pmatrix} z_1 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ z_2 & & & \\ & \ddots & \ddots & 0 \\ \vdots & & & \\ & & \ddots & \\ z_{\mu_i-\lambda_j} & & & z_1 \\ & \ddots & & \\ z_{\mu_i-\lambda_j+1} & & & z_2 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & & \\ & \ddots & \ddots & z_{\mu_i-\lambda_j} \\ 0 & \dots & 0 & z_{\mu_i-\lambda_j+1} \end{pmatrix} & ,\ \lambda_j \leq \mu_i \\ & \\ \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} & ,\ \lambda_j > \mu_i \end{cases}$$

tightness

A (C,A)-invariant subspace $\mathcal{V}$ is called *tight* if $\mathcal{V}+\operatorname{Ker}\mathsf{C}=\mathbb{R}^n$ or, equivalently, if $\operatorname{rk}\bar{C}=p$, i.e. $\lambda_p>0$.

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Then $\mathsf{C}Z\bar{C}^\top$ is invertible. We will consider only this case.

restricted output injection

The same construction can be repeated starting from a slightly different group action

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where U is unipotent lower triangular.

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The normal form (*Kronecker form*) looks very similar.

Kronecker form

$$\mathsf{C} = \left(\begin{array}{c|c|c} 0\ldots 0\,1 & & \\ \hline & \ddots & \\ \hline & & 0\ldots 0\,1 \\ \underbrace{\hspace{4em}}_{K_1} & \cdots & \underbrace{\hspace{4em}}_{K_p} \end{array}\right) \quad \textbf{and} \quad \mathsf{A} = \left(\begin{array}{c|c|c} \begin{matrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix} & & \\ \hline & \ddots & \\ \hline & & \begin{matrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix} \\ \underbrace{\hspace{6em}}_{K_1} & \cdots & \underbrace{\hspace{6em}}_{K_p} \end{array}\right)$$

but the *Kronecker indices* $(K_1, \ldots, K_p)$ are not neccessarily ordered. They are a permutation of the observability indices $(\mu_1, \ldots, \mu_p)$.

restrictions

Again we can look at restrictions of the pair (C, A) to a (C, A)-invariant subspace $\mathcal{V}$ and define the *restricted Kronecker indices* $(k_1, \ldots, k_p)$ which are a permutation of the restriction indices $(\lambda_1, \ldots, \lambda_p)$. It is $k_i \leq \mu_i$ for $i = 1, \ldots, p$.

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Again we get a characterisation of all bases of (C, A)-invariant subspaces.

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The codimension q tight (C, A)-invariant subspaces are precisely the images of full rank matrices $Z \in \mathbb{R}^{n\times(n-q)}$ which fullfill

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(2) $\mathsf{C}Z = \mathsf{C}Z\bar{C}^\top\bar{C}$

(3) $\mathsf{C}Z\bar{C}^\top$ **is unipotent**

where $(\bar{C}, \bar{A})$ is the matrix representation of a restriction in Kronecker form.

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The set of all these Zs is called $M(k,\mu)$, we denote $\Gamma(k) = M(k,k)$ which is a group. In fact $\operatorname{Im} Z_1 = \operatorname{Im} Z_2$ for $Z_1, Z_2 \in M(k,\mu)$ if and only if $Z_2 = Z_1 S^{-1}$ for $S \in \Gamma(k)$.

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Hence the *Kronecker stratum* of all codimension q conditioned invariant subspaces with fixed restricted Kronecker indices k is diffeomorphic to $M(k, \mu)/\Gamma(k)$ which is diffeomorphic to an $\mathbb{R}^N$.

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Hence the Brunovsky strata split in *Kronecker cells*.

generalised flag manifolds

Given a set of indices $0 \leq a_1 \leq \cdots \leq a_s = s$ the set

$$\mathrm{Flag}(a, \mathbb{R}^p) = \{(\mathcal{V}_1, \ldots, \mathcal{V}_s) \in \prod_{i=1}^{s} G_{a_i}(\mathbb{R}^p) \mid \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_s\}$$

of partial flags of *type* a is a compact analytic manifold.

generalised flag manifolds

Given a second set of indices $1 \leq b_1 \leq \cdots \leq b_s = p$ with $a_i \leq b_i$ for all $i = 1, \ldots, s$ we consider the *generalised flag manifold*

$$\mathrm{Flag}(a, b, \mathbb{R}^p) = \{(\mathcal{V}_1, \ldots, \mathcal{V}_s) \in \mathrm{Flag}(a, \mathbb{R}^p) \,| \\ \mathcal{V}_i \subset \mathrm{span}\{\mathrm{e}_1, \ldots, \mathrm{e}_{b_i}\}, i = 1, \ldots, s\}$$

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Here the indices a_i and b_i are the conjugate indices of λ_i and μ_i, respectively, read from right to left and aligned properly.

generalised flag manifolds

The manifold $\mathrm{Flag}(a, b, \mathbb{R}^p)$ is diffeomorphic to a quotient $V(a, b)/P(a)$ where $P(a) = V(a, a)$ is a parabolic group and $V(a, b)$ are the full rank $p \times p$ matrices where the last $p - b_i$ entries in the columns $a_{i-1} + 1, \ldots, a_i$ are zero.

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There is a cell decomposition of $\mathrm{Flag}(a, b, \mathbb{R}^p)$ with respect to each *reference flag* into the subsets with fixed intersection pattern with the reference flag. These cell decompositions are known as Bruhat decompositions.

the retraction map

Theorem [PH]:
The surjective smooth and closed maps

$$\gamma : M(\lambda, \mu) \longrightarrow V(a, b) \ ,$$
$$Z \mapsto \mathsf{C} Z \bar{C}^\top$$

and

$$\gamma : \Gamma(\lambda) \longrightarrow P(a) \ ,$$
$$S \mapsto \bar{C} S \bar{C}^\top$$

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$$\tilde{\gamma} : M(\lambda,\mu)/\Gamma(\lambda) \longrightarrow V(a,b)/P(a)$$

on quotients, in fact a deformation retract.

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Theorem:
$\tilde{\gamma}$ maps Kronecker cells onto Bruhat cells with respect to the reverse standard flag.